2.2.2 The case of nearly confluent poles

We consider the singular terms in the expansion in the nearly confluent case. That is, we suppose that ω is such that $\zeta_{j,0} \approx \zeta_{0,k}$ for some j and k. Our purpose is to write the singular expansion in a form that displays explicitly the limiting behavior if ω varies so that $\zeta_{j,0} \to \zeta_{0,k}$.

Let j and k be integers, and let $\alpha^* = k/j$, $\omega^* = \pi/\alpha^*$. Thus, $\zeta^*_{j,0} = j\alpha^* i = ki$ and $\zeta_{j,0}$ and $\zeta_{0,k}$ are simple poles if $\omega \neq \omega^*$ but $\omega - \omega^*$ is small. For $\bar{j} \neq j$ and $\bar{k} \neq k$, each of the functions $v_{\bar{j},0}$ and $v_{0,\bar{k}}$ is continuous in ω in a neighborhood of $\omega = \omega^*$. The functions $v_{j,0}$ and $v_{0,k}$ are not continuous at $\omega = \omega^*$, but the sum $S = v_{j,0} + v_{0,k}$ is a continuous function of ω in a neighborhood of $\omega = \omega^*$. We write $S = S_0 + S_1 + S_2$, where the three terms correspond to the contributions from g_0, g_1, and f respectively. Our computations will be carried out in three parts.

Calculation of S_0

From (II.2.2.1;5), (II.2.2.1;6), and (II.2.2.1;7), we have

$$S_0(x,\omega) = \Lambda'_j(g_0) r^{j\alpha} \sin j\alpha\theta + \frac{r^k \sin k(\omega-\theta)}{k! \sin k\omega} g_0^{(k)}(0).$$

From (II.2.2.1;21a) and the formulas preceding this formula,

$$\Lambda'_j(g_l) = -\frac{1}{\omega}\sum_{\mu=0}^{m-1} \frac{g_l^{(\mu)}(0)}{(j\alpha-\mu)\mu!} + \frac{1}{\omega}\int_0^\infty [g_l(x) - T_{l,m-1}(x)] x^{-1-j\alpha} dx,$$

where m is any integer with $j\alpha < m < s-1$. We suppose that $|\omega - \omega^*|$ is small enough so that $j\alpha < k+1$, and we select m so that $k+1 \le m < s-1$. With this restriction on m, $\mu = k$ is included in the sum, and we may write $S_0 = S_{0,1} + S_{0,2}$ where

$$S_{0,1}(x,\omega) = -\frac{1}{\omega}\frac{g_l^{(k)}(0)}{(j\alpha-k)k!} r^{j\alpha}\sin j\alpha\theta + \frac{r^k \sin k(\omega-\theta)}{k!\sin k\omega} g_0^{(k)}(0), \tag{1}$$

and

$$S_{0,2}(x,\omega) = \left\{ -\frac{1}{\omega}\sum_{\substack{\mu=0\\ \mu\neq k}}^{m-1} \frac{g_l^{(\mu)}(0)}{(j\alpha-\mu)\mu!} + \frac{1}{\omega}\int_0^\infty [g_l(x) - T_{l,m-1}(x)] x^{-1-j\alpha} dx \right\} r^{j\alpha}\sin j\alpha\theta. \tag{2}$$

As $\omega \to \omega^*$, the function $S_{0,2}(x,\omega)$ becomes the polynomial

$$S_{0,2}(x,\omega^*) = \left\{ -\frac{1}{\omega}\sum_{\substack{\mu=0\\ \mu\neq k}}^{m-1} \frac{g_l^{(\mu)}(0)}{(k-\mu)\mu!} + \frac{1}{\omega}\int_0^\infty [g_l(x) - T_{l,m-1}(x)] x^{-1-k} dx \right\} r^k \sin k\theta.$$

To study $S_{0,1}(x,\omega)$ we expand $\sin k(\omega-\theta) = \sin k\omega\cos k\theta - \cos k\omega \sin k\theta$ and write

$$S_{0,a}(x,\omega) = \frac{1}{k!} g_0^{(k)}(0) r^k \cos k\theta - \frac{1}{k!} G_0^{(k)}(0) \tilde{v}_{j,k}(x,\omega)$$

where

$$\tilde{v}_{j,k}(x,\omega) = \frac{r^{j\alpha}\sin j\alpha\theta}{\omega(j\alpha-k)} + \frac{r^k \cos k\omega \sin k\theta}{\sin k\omega}.$$

The function $\tilde{v}_{j,k}(x,\omega)$ is the linear combination of a singular function, $r^{j\alpha}\sin j\alpha\theta$, and a polynomial, $r^k \sin k\theta$. On the one hand, each of the coefficients in this linear combination becomes infinite as $\omega \to \omega^*$;

that is, as the pole $\zeta_{j,0} \to \zeta_{0,k}$. On the other hand, the function $\tilde{v}_{j,k}$ remains continuous in a neighborhood of $\omega = \omega^*$. Thus, in numerical computations with ω near to ω^*, the function $\tilde{v}_{j,k}$ is a good choice to use, for example, in enriching a finite element subspace.

For numerical computations, it is important to write $\tilde{v}_{j,k}$ in a form that avoids roundoff due to subtractive cancellation. For this, we note the formulas

$$j\alpha - k = \frac{k(\omega^* - \omega)}{\omega},\ k\omega = j\pi - k(\omega^* - \omega).$$

Since

$$\sin k\omega = (-1)^{j+1} \sin k(\omega^* - \omega),\ \cos k\omega = (-1)^j \cos k(\omega^* - \omega),$$

we have

$$\begin{aligned}\tilde{v}_{j,k}(x, \omega) =& \frac{r^{j\alpha} \sin j\alpha\theta}{k(\omega^* - \omega)} - r^k \cot k(\omega^* - \omega) \sin k\theta \\ =& A_1(x, \omega) + A_2(x, \omega) + A_3(x, \omega).\end{aligned}$$

where

$$\begin{aligned}A_1(x, \omega) =& \frac{r^{j\alpha} - r^k}{k(\omega^* - \omega} \sin j\theta \\ =& \frac{1 - r^{-k(\omega^* - \omega)/\omega}}{k(\omega^* - \omega} r^{j\alpha} \sin j\alpha\theta, \\ A_2(x, \omega) =& \frac{r^k (\sin j\alpha\theta - \sin k\theta)}{k(\omega^* - \omega)} \\ =& \frac{2r^k \cos(\frac{1}{2}k + \frac{1}{2}k(\omega^*/\omega))\theta \sin \frac{k(\omega^* - \omega)}{2\omega}\theta}{k(\omega^* - \omega)}, \\ A_3(x, \omega) =& r^k \sin k\theta \big[\frac{1}{k(\omega^* - \omega)} - \cot k(\omega^* - \omega)\big].\end{aligned}$$

Each of the functions A_1, A_2, and A_3, is by inspection, continuous at $\omega = \omega^*$, and a computation can be devised to evaluate each function accurately for small $\omega^* - \omega$. We therefore have obtained the desired expression for the singular function contained in S_0 in the case of nearly confluent poles. The singular functions contained in S_1 and S_2 are obtained in the same way.