2.2.1 Formulas for the singular functions and linear functionals

In the representation $u = V + w$ of the solution u of the boundary value problem (II.2.2;1), the function w is smooth and vanishes to a certain order at the origin. The function V arises from the residues coming from the poles in a contour integration, and is a sum of terms, one for each pole in each of the integrands appearing in (II.2.2;4). Each of these terms consists of a coefficient, which is a bounded linear functional of the data of the boundary value problem (II.2.2;1) times a function, which is independent of the data of the problem. The function either has a certain singularity at the origin, or is a polynomial of a certain degree. We now derive formulas for these linear functionals and functions.

As was mentioned in §II.2.2, the poles of the integrands in (II.2.2;4) that lie within the region of the contour integration consist of the points $\zeta_{j,0} = j\alpha i$, $j = 1, \cdots, J(s)$, and the points $\zeta_{0,k} = ki$, $k = 0, \cdots, n-1$. Here, the number s is determined by the regularity of the data of the problem (II.2.2;1), $n = \lfloor s \rfloor$ is the integer part of s, and $J(s) = \max\{j : j\alpha < s - 1\}$. The poles all lie on the imaginary axis. The points $\zeta_{j,0}$ are zeros of $\sinh \zeta\omega$, and in some sense are intrinsic to the boundary value problem. These poles give rise to singular terms in the function V. The points $\zeta_{0,k}$ are poles of the functions $\hat{f}$, $\hat{g}_0$, and $\hat{g}_1$. Thus, these poles are determined by the data of the problem, specifically, the degree of non-vanishing of the data at the apex of the sector. The poles $\zeta_{0,k}$ give rise to polynomial terms in the function V. It may happen that $\zeta_{j,0} = \zeta_{0,k}$ for some j and k. In this case, which we refer to as the confluent case, the formulas for the singular functions are more complicated.

To calculate the function $V(x, s-1)$, we write

$$V(x, s-1) = \sum_{1}^{J(s)} v_{j,0}(x) + \sum_{0}^{[s]-1} v_{0,k}(x),$$

where $v_{j,0}(x)$ is the contribution to V arising from the pole at $\zeta_{j,0}$, and $v_{0,k}(x)$ is the contribution to V arising from the pole at $\zeta_{0,k}$. The calculation is divided into the cases when the poles $\zeta_{j,0}$ and $\zeta_{0,k}$ are distinct, and the confluent case when $\zeta_{j,0} = \zeta_{0,k}$ for some j and k,

Calculation of $v_{j,0}$ *in the case of distinct poles*

To calculate the residues at the points $\zeta_{j,0}$ we use the formula $\sinh \zeta\omega \approx (-1)^j \omega(\zeta - j\alpha i)$ near $\zeta_{j,0}$. We obtain

$$\lim_{\zeta \to \zeta_{j,0}} (\zeta - \zeta_{j,0}) \hat{f}(\zeta - 2i, \varphi) \frac{\sinh \zeta(\omega - \theta) \sinh \zeta\varphi}{\zeta \sinh \zeta\omega} e^{i\zeta\tau} = \frac{(-1)^j i}{j\pi} \hat{f}((j\alpha - 2)i, \varphi) \sin j\alpha(\omega - \theta) \sin j\alpha\varphi e^{-j\alpha\tau},$$

$$\lim_{\zeta \to \zeta_{j,0}} (\zeta - \zeta_{j,0}) \hat{f}(\zeta - 2i, \varphi) \frac{\sinh \zeta\theta \sinh \zeta(\omega - \varphi)}{\zeta \sinh \zeta\omega} e^{i\zeta\tau} = \frac{(-1)^j i}{j\pi} \hat{f}((j\alpha - 2)i, \varphi) \sin j\alpha\theta \sin j\alpha(\omega - \varphi) e^{-j\alpha\tau},$$

$$\lim_{\zeta \to \zeta_{j,0}} (\zeta - \zeta_{j,0}) \hat{g}_0(\zeta) \frac{\sinh \zeta(\omega - \theta)}{\sinh \zeta\omega} e^{i\zeta\tau} = (-1)^j \frac{i}{\omega} \hat{g}_0(j\alpha i) \sin j\alpha(\omega - \theta) e^{-j\alpha\tau},$$

$$\lim_{\zeta \to \zeta_{j,0}} (\zeta - \zeta_{j,0}) \hat{g}_1(\zeta) \frac{\sinh \zeta\theta}{\sinh \zeta\omega} e^{i\zeta\tau} = (-1)^j \frac{i}{\omega} \hat{g}_1(j\alpha i) \sin j\alpha\theta e^{-j\alpha\tau},$$

The function $v^*_{j,0}$ is calculated from the formula

$$v^*_{j,0} = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \cdot \lim_{\zeta \to \zeta_{j,0}} = \sqrt{2\pi} i \cdot \lim_{\zeta \to \zeta_{j,0}} .$$

Using this and the fact that $\sin j\alpha(\omega-\theta) = (-1)^{j+1}\sin j\alpha\theta$, we obtain

$$\begin{aligned} v_{j,0}(x) =& \frac{1}{j}\sqrt{\frac{2}{\pi}} r^{j\alpha}\sin j\alpha\theta \int_0^\omega \hat{f}((j\alpha-2)i,\varphi)\sin j\alpha\varphi d\varphi \\ &+ \frac{\sqrt{2\pi}}{\omega} r^{j\alpha}\sin j\alpha\theta[\hat{g}_0(j\alpha i) + (-1)^{j+1}\hat{g}_1(j\alpha i)] \\ =& \Lambda_j\{f, g_0, g_1\} r^{j\alpha}\sin j\alpha\theta, \end{aligned} \tag{1}$$

where

$$\Lambda_j\{f, g_0, g_1\} = \Lambda_j'(g_0) + (-1)^{j+1}\Lambda_j'(g_1) + \Lambda_j''(f), \tag{2a}$$

and where the linear functionals Λ_j' and Λ_j'' are given by

$$\begin{aligned} \Lambda_j'(g) &= \frac{\sqrt{2\pi}}{\omega}\hat{g}(j\alpha i), \\ \Lambda_j''(f) &= \frac{1}{j}\sqrt{\frac{2}{\pi}}\int_0^\omega \hat{f}((j\alpha-2)i,\varphi)\sin j\alpha\varphi d\varphi. \end{aligned} \tag{2b}$$

Calculation of $v_{0,k}$ *in the case of distinct poles*

To calculate the residues at the points $\zeta_{0,k}$, we use the formulas

$$\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi) = \frac{-i}{\sqrt{2\pi}}\sum_{l=0}^{k-2}\frac{\cos^l\varphi\sin^{k-2-l}\varphi}{l!(k-2-l)!}[D_{x_1}^l D_{x_2}^{k-2-l} f(0,0)],$$

$$\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{g}_l(\zeta) = \frac{-i}{\sqrt{2\pi}}\frac{1}{k!}g_l^{(k)}(0).$$

We therefore obtain

$$\begin{aligned} &\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi)\frac{\sinh\zeta(\omega-\theta)\sinh\zeta\varphi}{\zeta\sinh\zeta\omega}e^{i\zeta\tau} \\ &\qquad = \frac{-i}{\sqrt{2\pi}}\frac{\sin k(\omega-\theta)\sin k\varphi}{k\sin k\omega}e^{-k\tau}\sum_{l=0}^{k-2}\frac{\cos^l\varphi\sin^{k-2-l}\varphi}{l!(k-2-l)!}[D_{x_1}^l D_{x_2}^{k-2-l}f(0,0)], \end{aligned}$$

$$\begin{aligned} &\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi)\frac{\sinh\zeta\theta\sinh\zeta(\omega-\varphi)}{\zeta\sinh\zeta\omega}e^{i\zeta\tau} \\ &\qquad = \frac{-i}{\sqrt{2\pi}}\frac{\sin k\theta\sin k(\omega-\varphi)}{k\sin k\omega}e^{-k\tau}\sum_{l=0}^{k-2}\frac{\cos^l\varphi\sin^{k-2-l}\varphi}{l!(k-2-l)!}[D_{x_1}^l D_{x_2}^{k-2-l}f(0,0)], \end{aligned}$$

$$\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{g}_0(\zeta)\frac{\sinh\zeta(\omega-\theta)}{\sinh\zeta\omega}e^{i\zeta\tau} = \frac{-i}{\sqrt{2\pi}}\frac{1}{k!}\frac{\sin k(\omega-\theta)}{\sin k\omega}e^{-k\tau}g_0^{(k)}(0),$$

$$\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{g}_1(\zeta)\frac{\sinh\zeta\theta}{\sinh\zeta\omega}e^{i\zeta\tau} = \frac{-i}{\sqrt{2\pi}}\frac{1}{k!}\frac{\sin k\theta}{\sin k\omega}e^{-k\tau}g_1^{(k)}(0).$$

The function $v_{0,k}^*$ is calculated from the formula

$$v_{0,k}^* = \frac{1}{\sqrt{2\pi}}\cdot 2\pi i\cdot\lim_{\zeta\to\zeta_{0,k}} = \sqrt{2\pi}i\cdot\lim_{\zeta\to\zeta_{0,k}}.$$

Since $e^{-k\tau} = r^k$, we therefore obtain

$$\begin{aligned} v_{0,k}(x) = &\frac{r^k \sin k(\omega-\theta)}{k \sin k\omega} \sum_{l=0}^{k-2} \frac{D^l_{x_1} D^{k-2-l}_{x_2} f(0,0)}{l!(k-2-l)!} \int_0^\theta \sin k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi \\ &+ \frac{r^k \sin k\theta}{k \sin k\omega} \sum_{l=0}^{k-2} \frac{D^l_{x_1} D^{k-2-l}_{x_2} f(0,0)}{l!(k-2-l)!} \int_\theta^\omega \sin k(\omega-\varphi) \cos^l \varphi \sin^{k-2-l} \varphi d\varphi \\ &+ \frac{r^k \sin k(\omega-\theta)}{k! \sin k\omega} g_0^{(k)}(0) + \frac{r^k \sin k\theta}{k! \sin k\omega} g_1^{(k)}(0). \end{aligned}$$

Evidently the coefficients of $g_0^{(k)}(0)$ and $g_1^{(k)}(0)$ are homogeneous polynomials of degree k in x. We now assert that

$$\begin{aligned} P(x) = &r^k \sin k(\omega-\theta) \int_0^\theta \sin k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi \\ &+ r^k \sin k\theta \int_\theta^\omega \sin k(\omega-\varphi) \cos^l \varphi \sin^{k-2-l} \varphi d\varphi \end{aligned}$$

is a homogeneous polynomial of degree k in x. Since $\cos^l \varphi \sin^{k-2-l} \varphi$ is a trigonometric polynomial of degree $k-2$, and is therefore a linear combination of polynomials of the form $\cos((k-2)\varphi + \alpha)$, it suffices to show that

$$\begin{aligned} P_1(x) = &r^k \sin k(\omega-\theta) \int_0^\theta \sin k\varphi \cos((k-2)\varphi+\alpha) d\varphi \\ &+ r^k \sin k\theta \int_\theta^\omega \sin k(\omega-\varphi) \cos((k-2)\varphi+\alpha) d\varphi \end{aligned}$$

is a polynomial in x. Writing

$$\begin{aligned} \sin k\varphi \cos((k-2)\varphi+\alpha) &= \tfrac{1}{2} \sin((2k-2)\varphi+\alpha) + \tfrac{1}{2} \sin(2\varphi-\alpha), \\ \sin k(\omega-\varphi) \cos((k-2)\varphi+\alpha) &= \tfrac{1}{2} \sin(k\omega - 2\varphi + \alpha) + \tfrac{1}{2} \sin(k\omega - (2k-2)\varphi - \alpha) \end{aligned}$$

and evaluating the integrals, we obtain $P_1 = P_2 - P_3$ where

$$\begin{aligned} P_2(x) = &C_1 r^k \sin k(\omega-\theta) + C_2 r^k \sin k\theta, \\ P_3(x) = &\frac{1}{4} r^k \big(\sin k(\omega-\theta) \cos(2\theta-\alpha) + \sin k\theta \cos(k\omega - 2\theta + \alpha) \big) \\ &+ \frac{1}{4(k-1)} r^k \big(\sin k(\omega-\theta) \cos((2k-2)+\alpha) + \sin k\theta \cos(k\omega - (2k-2)\theta - \alpha) \big). \end{aligned}$$

Here C_1 and C_2 are constants independent of θ. Evidently $P_2(x)$ is a polynomial of degree k in x. Since

$$\begin{aligned} &\sin k(\omega-\theta) \cos(2\theta-\alpha) + \sin k\theta \cos(k\omega - 2\theta + \alpha) \\ &\qquad = \tfrac{1}{2} \sin((k-2)\theta + k\omega + \alpha) - \tfrac{1}{2} \sin((k-2)\theta - k\omega + \alpha) \\ &\sin k(\omega-\theta) \cos((2k-2)\theta+\alpha) + \sin k\theta \cos(k\omega - (2k-2)\theta - \alpha) \\ &\qquad = \tfrac{1}{2} \sin((k-2)\theta + k\omega + \alpha) - \tfrac{1}{2} \sin((k-2)\theta - k\omega + \alpha) \end{aligned}$$

we see that $P_3(x)$ is a polynomial in x of degree k. We conclude that

$$v_{0,k}(x) = \sum_{l=0}^{k-2} [D^l_{x_1} D^{k-2-l}_{x_2} f(0,0)] P_{k,l}(x) + g_0^{(k)}(0) P_k^{(0)}(x) + g_1^{(k)}(0) P_k^{(1)}(x), \tag{3}$$

where $P_{k,l}$, $P_k^{(0)}$, and $P_k^{(1)}$ are homogeneous polynomials in x of degree k.

Calculation of $v_{j,0}$ in the confluent case $\zeta_{j,0} = \zeta_{0,k}$.

We suppose that $j\alpha = k$ for some integers j and k. In this case, the integrand has a double pole at $\zeta_{j,0}$. To calculate the residues of the four integrals appearing in (II.2.2;4), we use the following simple fact. Let $\beta(\zeta)$ and $\gamma(\zeta)$ be meromorphic near $\zeta = 0$, and let $\delta(\zeta)$ be analytic near $\zeta = 0$, with expansions

$$\beta(\zeta) = \beta_{-1}\zeta^{-1} + \beta_0 + \beta_1\zeta + \cdots,$$

$$\gamma(\zeta) = \gamma_{-1}\zeta^{-1} + \gamma_0 + \gamma_1\zeta + \cdots,$$

$$\delta(\zeta) = \delta_0 + \delta_1\zeta + \delta_2\zeta^2 + \cdots.$$

Setting $\mu = \beta\gamma\delta$, we have

$$\mu(\zeta) = \mu_{-2}\zeta^{-1} + \mu_{-1}\zeta^{-1} + \mu_0 + \cdots,$$

where

$$\mu_{-1} = \beta_{-1}\gamma_0\delta_0 + \beta_0\gamma_{-1}\delta_0 + \beta_{-1}\gamma_{-1}\delta_1. \tag{4}$$

We first apply (4) with

$$\beta(\zeta) = \hat{f}(\zeta - 2i, \varphi), \tag{5a}$$

$$\gamma(\zeta) = \frac{1}{\zeta \sinh \zeta\omega}, \tag{5b}$$

$$\delta(\zeta) = e^{i\zeta\tau} \sinh \zeta(\omega - \theta) \sinh \zeta\varphi, \tag{5c}$$

and with expansions at $\zeta = \zeta_{j,0}$ instead of expansions at $\zeta = 0$. This choice enables us to calculate the contribution to the first integral on the right side of (II.2.2;4) arising from the confluent pole $\zeta_{j,0} = \zeta_{0,k}$. Notice that $\delta_0 = -r^k \sin k(\omega - \theta) \sin k\varphi$. Using this fact, one sees after a computation that the contributions to the first integral in (II.2.2;4) coming from the first two terms in (4) are homogeneous polynomials in x of degree k. These terms do not contribute to the singular functions arising from this pole. To determine the contribution to the integral coming from the term $\beta_{-1}\gamma_{-1}\delta_1$, we calculate

$$\begin{aligned}\beta_{-1} &= \lim_{\zeta \to ki} (\zeta - ki)\hat{f}(\zeta - 2i, \varphi)\\ &= \frac{-i}{\sqrt{2\pi}} \sum_{l=0}^{k-2} \frac{\cos^l \varphi \sin^{k-2-l} \varphi}{l!(k-2-l)!} [D_{x_1}^l D_{x_2}^{k-2-l} f(0,0)],\end{aligned}$$

$$\gamma_{-1} = \lim_{\zeta \to \zeta_{j,0}} \frac{\zeta - j\alpha i}{\zeta \sinh \zeta\omega} = \frac{(-1)^{j+1} i}{j\pi},$$

$$\delta'(\zeta) = e^{i\zeta\tau} \big(i\tau \sinh \zeta(\omega - \theta) \sinh \zeta\varphi + (\omega - \theta) \cosh \zeta(\omega - \theta) \sinh \zeta\varphi + \varphi \sinh \zeta(\omega - \theta) \cosh \zeta\varphi \big),$$

$$\begin{aligned}\delta_1 =& \delta'(\zeta_{j,0}) = (-1)^{j+1} i r^k \big((\ln r) \sin k\theta \sin k\varphi + \theta \cos k\theta \sin k\varphi + \varphi \sin k\theta \cos k\varphi \big)\\ &+ \text{ a homogeneous polynomial in } x \text{ of degree } k.\end{aligned}$$

Using these formulas, we see that the contribution to the first integral in (II.2.2;4) is

$$\begin{aligned}&Ar^k \big[(\ln r) \sin k\theta \int_0^\theta \sin k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi + \theta \cos k\theta \int_0^\theta \sin k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi\\ &\qquad + \sin k\theta \int_0^\theta \varphi \cos k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi \big].\end{aligned} \tag{6}$$

where

$$A = \sum_{l=0}^{k-2} \bar{C}_{k,l} [D_{x_1}^l D_{x_2}^{k-2-l} f(0,0)].$$

Next, we apply the formula (4) with β and γ given by (5a,b), and with δ replaced by

$$\delta^*(\zeta) = e^{i\zeta\tau} \sinh \zeta\theta \sinh \zeta(\omega - \varphi), \tag{5c*}$$

This choice will enable us to calculate the contribution to the second integral on the right side of (II.2.2;4) arising from the confluent pole $\zeta_{j,0} = \zeta_{0,k}$. Notice that $\delta^*(\zeta)$ can be obtained from $\delta(\zeta)$ by interchanging θ and φ. Therefore, $\delta_1^* = \delta^{*\prime}(\zeta_{j,0})$ can be obtained from δ_1 by interchanging θ and φ. Using this, we see that the contribution to the second integral is

$$\begin{aligned} &Ar^k \big[(\ln r) \sin k\theta \int_\theta^\omega \sin k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi + \theta \cos k\theta \int_\theta^\omega \sin k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi \\ &\qquad + \sin k\theta \int_\theta^\omega \varphi \cos k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi\big]. \end{aligned} \tag{7}$$

Adding the expressions (6) and (7), we find that the contribution to the first two integrals in (II.2.2;4) is $\Lambda_j''(f)[(\ln r) r^k \sin k\theta + \theta r^k \cos k\theta]$, where

$$\Lambda_j''(f) = \sum_{l=0}^{k-2} C_{k,l} [D_{x_1}^l D_{x_2}^{k-2-l} f(0,0)],\ \ k = j\alpha = \text{integer}. \tag{8}$$

The coefficient $C_{k,l} = \bar{C}_{k,l} \int_0^\omega \sin k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi$.

Next we consider the residues coming from the third integral in (II.2.2;4). We apply (4) with

$$\beta(\zeta) = \hat{g}_0(\zeta),$$

$$\gamma(\zeta) = \frac{1}{\sinh \zeta\omega},$$

$$\delta(\zeta) = e^{i\zeta\tau} \sinh \zeta(\omega - \theta).$$

Again, since $\delta_0 = r^k \sin k\theta$, the terms in (4) that contain δ_0 lead to homogeneous polynomials in x of degree k. To determine the contribution to the integral coming from the term $\beta_{-1}\gamma_{-1}\delta_1$, we calculate

$$\beta_{-1} = \frac{-i}{\sqrt{2\pi}} \frac{1}{k!} g_0^{(k)}(0),$$

$$\gamma_{-1} = \lim_{\zeta \to \zeta_{j,0}} \frac{\zeta - j\alpha i}{\sinh \zeta\omega} = \frac{(-1)^j}{\omega},$$

$$\delta'(\zeta) = e^{i\zeta\tau} \big(i\tau \sinh \zeta(\omega - \theta) + (\omega - \theta) \cosh \zeta(\omega - \theta)\big),$$

$$\delta_1 = \delta'(\zeta_{j,0}) = (-1)^{j+1} [(\ln r) r^k \sin k\theta + \theta r^k \cos k\theta].$$

Using these formulas, we see that the contribution to the third integral in (II.2.2;4) is

$$\Lambda_j'(g_0)[(\ln r) r^k \sin k\theta + \theta r^k \cos k\theta],$$

where the linear functional $\Lambda_j'(g)$ is defined by

$$\Lambda_j'(g) = C_j g^{(k)}(0),\ \ k = j\alpha = \text{integer}. \tag{9}$$

A calculation shows that the analogous contribution to the fourth integral in (II.2.2;4) is

$$(-1)^{j+1}\Lambda_j'(g_1)[(\ln r)r^k \sin k\theta + \theta r^k \cos k\theta].$$

Combining these formulas, we have

$$v_{j,0}(x) = \Lambda_j\{f, g_0, g_1\}[(\ln r)r^k \sin k\theta + \theta r^k \cos k\theta] + v_{0,k}(x),\ \ k = j\alpha = \text{ integer}, \tag{10}$$

where $\Lambda_j\{f, g_0, g_1\}$ is defined by (2a) with Λ_j' and Λ_j'' given by (9) and (8), and where $v_{0,k}(x)$ is a homogeneous polynomial of degree k.

The following theorem is an immediate consequence of Theorem II.2.2;1 and the above formulas.

Theorem 1. *Suppose* $\{f, g_0, g_1\} \in \mathcal{Y}_D^s(S)$ *with* $s > 2$, $s \neq$ *integer,* $(s-1)/\alpha \neq$ *integer. Let* u *be the solution of (II.2.2;1), and suppose* $u \equiv 0$ *for* $r > 1$. *Then* $u(x) = V(x, s-1) + w(x, s-1)$ *where* w *is given by (II.2.2;4), where*

$$V(x, s-1) = \sum_{j=1}^{J(s)} v_{j,0}(x) + \sum_{\substack{k=1 \\ k/\alpha \neq \text{integer}}}^{[s]-1} v_{0,k}(x),$$

and where the functions $v_{j,0}$ *and* $v_{0,k}$ *are given by (1), (3), and (10). The functions* $V(x, s-1)$ *and* $w(x, s-1)$, *regarded as functions of* (x, ω) *with* $x \in S(\omega)$, *are jointly continuous in these variables. For each* $a > 0$, $w \in H^s(S_a)$ *and*

$$||w(\cdot, s-1)||_{H^s(S_a)} \leq C(a)||\{f, g_0, g_1\}||_{\mathcal{Y}_D^s(S)}, \tag{11}$$

where S_a *is the truncated sector of radius* a. *Also,*

$$D_x^m w(0, s-1) = 0 \text{ for } 0 \leq m \leq [s] - 2.$$

Finally, $u \in H^s(S)$ *if and only if either* $J(s) = 0$ *or* $\Lambda_j\{f, g_0, g_1\} = 0$ *for* $j = 1, \cdots, J(s)$ *and, if* $j\alpha = k =$ *integer,* $D^{k-2}f(0,0) = 0$, $g_0^{(k)}(0) = g_1^{(k)}(0) = 0$.

In Theorem 1, the hypothesis $(s-1)/\alpha \neq$ integer is needed to prevent a pole of csch $\alpha\omega$ from lying on the line of integration. This hypothesis probably cannot be removed. The hypothesis $s \neq$ integer is needed because $\hat{f}(\zeta - 2i, \varphi)$ and $\hat{g}_l(\zeta)$ may have poles when $\Im\zeta =$ integer, so $w^*(\tau, \theta, s-1)$ may not be defined when $s =$ integer. We know that the singularities in the solution u do not arise from the poles $\zeta_{0,k}$, unless they are confluent poles. It is natural to seek a decomposition of the solution in the case $\{f, g_0, g_1\} \in \mathcal{Y}_D^s(S)$ for $s =$ integer. This is done in the next theorem.

Let

$$v_j(x) = \begin{cases} r^{j\alpha} \sin j\alpha\theta, & j\alpha \neq \text{integer}, \\ (\ln r)r^k \sin k\theta + \theta r^k \cos k\theta, & j\alpha = k = \text{integer}, \end{cases} \tag{12}$$

$$v(x, s-1) = \sum_{1}^{J(s)} \Lambda_j\{f, g_0, g_1\} v_{j,0}(x), \tag{13}$$

$$P(x, s-1) = \sum_{k=1}^{[s]-1} v_{0,k}(x). \tag{14}$$

Thus, we have

$$V(x, s-1) = v(x, s-1) + P(x, s-1),$$

and we define

$$W(x, s-1) = w(x, s-1) + P(x, s-1).$$

Using these functions, there is obtained two representations of u:

$$\begin{aligned} u(x) &= v(x, s-1) + P(x, s-1) + w(x, s-1) \\ &= v(x, s-1) + W(x, s-1) \qquad \text{(15a)} \\ &= V(x, s-1) + w(x, s-1). \qquad \text{(15b)} \end{aligned}$$

The properties of the representation (15b) are established in Theorem 1, and the properties of (15a) are established in Theorem 2 below. Each of the representations (15a) and (15b) have advantages and disadvantages. The principal advantage of the representation (15a) lies in the fact, established in Theorem 2, that this representation is also valid when $s =$ integer. Also, (15a) clearly displays the fact that $u \in H^s(S)$ if and only if all the linear functionals $\Lambda_j\{f, g_0, g_1\} = 0$. On the other hand, the linear functionals Λ_j become infinite when ω is varied so that $j\alpha \to$ integer, and the function $v(x, s-1)$ is not continuous as a function of ω. The representation (19b) remedies this defect. The function $V = v + P$ is continuous in ω, the singularity in v being balanced by a singularity in P. This advantage comes at the cost of a greater complexity in the singular expansion V.

Theorem 2. *Suppose* $\{f, g_0, g_1\} \in \mathcal{Y}^s_D(S)$ *with* $s \geq 2$, $(s-1)/\alpha \neq$ *integer. Let* u *be the solution of (II.2.2;1), and suppose* $u \equiv 0$ *for* $r > 1$. *Then*

$$u(x) = \sum_{1}^{J(s)} \Lambda_j\{f, g_0, g_1\} v_j(x) + W(x, s-1),$$

where $\Lambda_j\{f, g_0, g_1\}$, $j = 1, \cdots, J(s)$ *are bounded linear functionals on* $\mathcal{Y}^s_D(S)$, v_j *is given by (1), and, for each* $a > 0$, $W \in H^s(S_a)$. *There is a constant* $C(a) > 0$ *such that*

$$||W(\cdot, s-1)||_{H^s(S_a)} \leq C(a) ||\{f, g_0, g_1\}||_{\mathcal{Y}^s_D(S)}. \tag{16}$$

Proof. If $s > 2$ and $s \neq$ integer, the result follows from Theorem 1. Suppose $s = k \geq 2$ is an integer, and $(s-1)/\alpha \neq$ integer. Let $\varepsilon > 0$ be such that $j\alpha \neq (k-\varepsilon, k+\varepsilon)$ for $j = 1, \cdots$. Thus, $J(s-\varepsilon) = J(s+\varepsilon)$, so $v(x, k-\varepsilon-1) = v(k+\varepsilon-1)$, and therefore, $W(x, k-\varepsilon-1) = W(x, k+\varepsilon-1)$. We define $W(x, k-1) = W(x, k-\varepsilon-1)$. From Theorem 1, for each $a > 0$,

$$||W(\cdot, k-\varepsilon-1)||_{H^{k-\varepsilon}(S_a)} \leq C(a) ||\{f, g_0, g_1\}||_{\mathcal{Y}^{k-\varepsilon}_D(S)},$$

$$||W(\cdot, k+\varepsilon-1)||_{H^{k+\varepsilon}(S_a)} \leq C(a) ||\{f, g_0, g_1\}||_{\mathcal{Y}^{k+\varepsilon}_D(S)}.$$

Hence, by interpolation,

$$||W(\cdot, k-1)||_{H^k(S_a)} \leq C(a) ||\{f, g_0, g_1\}||_{\mathcal{Y}^k_D(S)}.$$

If $2 < s \leq 1 + \alpha$, $J(s) = 0$, $u = W(\cdot, s-1)$, and the bound for u follows from (11). In the case $s = 2$, the proof is a little different. Let $\{f, g_0, g_1\} \in \mathcal{Y}^2_D(S)$, let $\varepsilon > 0$, and let $\{f^\mu, g^\mu_0, g^\mu_1\} \in \mathcal{Y}^{2+\varepsilon}_D(S)$ be a sequence with $||\{f^\mu - f, g^\mu_0 - g_0, g^\mu_1 - g_1\}||_{\mathcal{Y}^2_D(S)} \to 0$ as $\mu \to \infty$. Let u^μ be the solution of (1) with data $\{f^\mu, g^\mu_0, g^\mu_1\}$.

Suppose, without loss of generality, that $u^\mu(x) = 0$ for $r > 1$. Suppose ε is so small that $2+\varepsilon < (J(2)+1)\alpha$, so $J(2+\varepsilon) = J(2)$. Applying Theorem 2 with $s = 2+\varepsilon$, we have the decomposition

$$u^\mu(x) = v^\mu(x, s+\varepsilon-1) + W^\mu(x, s+\varepsilon-1).$$

The singular expansion v^μ is given by

$$v^\mu(x, s+\varepsilon-1) = \sum_1^{J(2)} \Lambda_j\{f^\mu, g_0^\mu, g_1^\mu\} v_j(x).$$

Since each Λ_j is a bounded linear functional on $\mathcal{Y}_D^2(S)$, $v^\mu \to v$ as $\mu \to \infty$ where

$$v(x,1) = \sum_1^{J(2)} \Lambda_j\{f, g_0, g_1\} r^{j\alpha} \sin j\alpha\theta.$$

Since $\Delta v^\mu = 0$ and $v^\mu = 0$ on Γ_0 and Γ_1, $\Delta W^\mu = f^\mu$ and $W^\mu = g_l^\mu$ on Γ_l, $l = 0, 1$. From Theorem 2 in the case $s > 2$, $W^\mu \in H^{2+\varepsilon}(S_a)$. Hence we may apply (2.1;25) to $W^\mu - W^\nu$ to obtain $||W^\mu - W^\nu||_{H^2(S)} \to 0$ as $\mu, \nu \to \infty$. Hence W^μ converges in $H^2(S)$ to a function $W \in H^2(S)$. We therefore obtain the desired decomposition of u in the case $s = 2$. ■

It may be noted that when α is irrational, the singular functions $v_j = r^{j\alpha} \sin j\alpha\theta$ are orthogonal on the arc $r =$ const., and in fact are exactly the functions used in a separation of variables of the Laplace operator in the sector S. When α is rational, and $j\alpha$ is an integer, $r^{j\alpha} \sin j\alpha\theta$ is a polynomial and thus is not singular. As the formula (12) shows, the appropriate singular function no longer occurs in the separation of variables solution.

In Theorems 1 and 2 it is assumed that $(s-1)/\alpha \neq$ integer. It is useful to give a reformulation of Theorem 2 that highlights these exceptional values. Let $s_j = j\alpha + 1$. Thus, $(s_j - 1)/\alpha = j$, so the s_j are the exceptional values. The following follows directly from Theorem 2.

Theorem 3. *There are linear functionals Λ_j and functions v_j, $j = 1, 2, \cdots$, with the following properties. (i) Λ_j is a bounded linear functional on $\mathcal{Y}_D^s$ for $s > s_j$, but not for $s \leq s_j$. (ii) $v_j \in H^s(S) \cap H_0^1(S)$ for $s < s_j$ but not for $s \geq s_j$. Also v_j is smooth everywhere except at the origin and $\Delta v_j = 0$ in a neighborhood of the origin. (iii) If $s_j < s < s_{j+1}$, $\{f, g_0, g_1\} \in \mathcal{Y}_D^s$, u is the solution of (II.2.2;1), and $u \equiv 0$ outside S_1, then for each $a > 0$,*

$$u_j := u - \sum_{l=1}^{j} \Lambda_l\{f, g_0, g_1\} v_l \in H^s(S_a), \quad \textit{with } \|u_j\|_{H^s(S_a} \leq C(a) \|\{f, g_0, g_1\}\|_{\mathcal{Y}_D^s}.$$

Calculation of Λ_j in the case of distinct poles

In the confluent case, when $j\alpha = k$ for some integers j and k, the linear functionals $\Lambda_j'(g_l)$ and $\Lambda_j''(f)$ are given by (9) and (8). It is of interest to obtain formulas in the case $j\alpha \neq$ integer. For this, we must evaluate $\hat{f}(\eta i, \theta)$ for $\eta > -1$ and $\hat{g}_l(\eta i)$ for $\eta > 0$. We have

$$\begin{aligned} \hat{g}_l(\eta i) =& \hat{T}_{l,m-1}(\eta i) + \frac{1}{\sqrt{2\pi}} \int_0^\infty R_{l,m-1}^*(\tau) e^{\eta\tau} d\tau \\ =& \frac{-1}{\sqrt{2\pi}} \sum_0^{m-1} \frac{g_l^{(k)}(0)}{(\eta-k)k!} + \frac{1}{\sqrt{2\pi}} \int_0^\infty [g_l(x) - T_{l,m-1}(x)] x^{-\eta-1} dx, \end{aligned}$$

$$
\begin{aligned}
\hat{f}(\eta i,\theta) =&\hat{T}_{m-3}(\eta i,\theta)+\frac{1}{\sqrt{2\pi}}\int_0^\infty R_m^*(\tau,\theta)e^{\eta\tau}d\tau\\
=&\frac{-1}{\sqrt{2\pi}}\sum_{0\le k+l\le m-3}\frac{\cos^k\theta\sin^l\theta}{k!l!(\eta-(k+l))}[D_{x_1}^kD_{x_2}^lf(0,0)]\\
&+\frac{1}{\sqrt{2\pi}}\int_0^\infty[f(r\cos\theta,r\sin\theta)-T_{m-3}(r\cos\theta,r\sin\theta)]r^{-\eta-1}dr.
\end{aligned}
$$

Since $\{f,g_0,g_1\}\in\mathcal{Y}_D^s(S)$, these formulas are valid for any integer m which satisfies $\eta<m<s-1$. To apply (2b), we must evaluate these formulas at $\eta=j\alpha$. We choose $m=[j\alpha]+1$ to obtain

$$
\Lambda_j'(g_l)=-\frac{1}{\omega}\sum_0^{[j\alpha]}\frac{g_l^{(k)}(0)}{(j\alpha-k)k!}+\frac{1}{\omega}\int_0^\infty[g_l(x)-T_{l,[j\alpha]}(x)]x^{-1-j\alpha}dx,\ j\alpha\ne\text{integer},\tag{17a}
$$

$$
\begin{aligned}
\Lambda_j''(f)=&-\frac{1}{j\pi}\sum_{0\le k+l\le[j\alpha]-2}\frac{A_{k,l}(\omega)}{k!l!(j\alpha-(k+l+2))}[D_{x_1}^kD_{x_2}^lf(0,0)]\\
&+\frac{1}{j\pi}\int\int_S[f(x)-T_{[j\alpha]-2}(x)]r^{-j\alpha}\sin j\alpha\theta dx,
\end{aligned}
\qquad j\alpha\ne\text{integer},\tag{17b}
$$

where

$$
A_{k,l}(\omega)=\int_0^\omega\cos^k\varphi\sin^l\varphi\sin j\alpha\varphi d\varphi.\tag{17c}
$$

From the construction, if $j\alpha\ne$ integer the mappings $\{f,g_0,g_1\}\mapsto\Lambda_j'(g_l)$ and $\{f,g_0,g_1\}\to\Lambda_j''(f)$ are bounded linear functionals on $\mathcal{Y}_D^s(S)$. We verify this directly. Since $g_l\in H^{s-1/2}(R_+)$ and $s=n+\sigma$ with $0<\sigma<1$, g_l is $(n-1)$-times continuously differentiable on R_+, so $g_l\to g_l^{(k)}$ is a bounded linear functional on $H^{s-1/2}(R_+)$ if $k\le n-1$. Also, from Theorem I.2;6,

$$
\int_0^1x^{-2(s-1/2)}|g_l-T_{l,n-1}|^2dx\le C||g_l||_{s-1/2}.
$$

Since $j\alpha<s-1$,

$$
\begin{aligned}
|\int_0^1[g_l(x)-T_{l,n-1}(x)]x^{-1-j\alpha}dx|\le&\int_0^1x^{-(s-1/2)}|g_l(x)-T_{l,n-1}(x)|\cdot x^{-1-j\alpha+s-1/2}dx\\
\le&\Big(\int_0^1x^{2(s-j\alpha-3/2)}dx\Big)^{1/2}||g_l||_{s-1/2}\\
\le&C||g_l||_{s-1/2}.
\end{aligned}
$$

Hence $\Lambda_j'(g_l)$ is a bounded linear functional on $H^{s-1/2}(R_+)$, and so is a bounded linear functional on $\mathcal{Y}_D^s(S)$. Similarly, since $f\in H^{s-2}(S)$, f has $n-3$ continuous derivatives, provided $s>3$. Hence $f\to D_{x_1}^kD_{x_2}^mf(0,0)$ is a bounded linear functional on $\mathcal{Y}_D^s(S)$ for $k+m\le n-3$, provided $s>3$. Also, from Theorem I.4;6,

$$
\int\int_Sr^{-2(s-2)}[f-T_{n-3}]^2dx\le C||f||_{s-2}.
$$

Since $j\alpha<s-1$,

$$
\begin{aligned}
|\int\int_S[f-T_{n-3}]r^{-j\alpha}\sin j\alpha\varphi dx|\le&\int\int_Sr^{-(s-2)}|f-T_{n-3}|\cdot r^{-j\alpha+s-2}dx\\
\le&C\Big(\int\int_Sr^{2(-j\alpha+s-2)}dx\Big)^{1/2}||f||_{s-2}\\
\le&C||f||_{s-2}.
\end{aligned}
$$

Hence $\Lambda_j''(f)$ is a bounded linear functional on $\mathcal{Y}_D^s(S)$.

From the formulas, it is seen that $\Lambda_j\{f,g_0,g_1\}$, and hence, $v_{j,0}$, becomes singular if $j\alpha\to$ integer. This does not contradict the assertion in Theorem 1 that v is a continuous function of ω, because the singularity in $v_{j,0}$ that appears when $j\alpha\to$ integer is cancelled by a singularity of opposite sign in $v_{0,k}$.