2. Expansion of the solution in a sector – preliminary results

In the theory of elliptic equations with smooth coefficients, an important tool is the analysis of equations with constant coefficients in a half-plane. When the Fourier transform is taken, the resulting system of ordinary differential equations can be solved exactly, and important features of the original problem can be found in the solution to this transformed problem. The study of an elliptic equation in a sector plays a similar role in the theory of piecewise smooth elliptic problems. The Kondrat'yev transform yields an ordinary differential equation whose solution shows the corner singularities that are present in the primal problem. In §II.1, the transformed equation was derived, and §§II.2.2-II.2.4, contains the most important properties of the transformed problem: the analytic continuation with respect to the parameter ζ. In this section, the analytic continuation is presented in a simplified way for Dirichlet boundary conditions. The simplification comes in not making minimal assumptions on the regularity of the data. Instead of requiring the data to belong to certain Sobolev spaces, it is merely required that they are continuously differentiable up to a certain order. The resulting expansions are somewhat simpler to derive, and it is hoped that the derivation will provide a useful preparation for the following sections.

Let S be the sector of angle ω defined, in terms of polar coordinates, by $S = \{(x, y) : 0 < \theta < \omega\}$. Let $\Gamma = \Gamma_0 \cup \Gamma_1$ be the boundary of S, where Γ_0 and Γ_1 are the two sides of S. We shall also consider the truncated sector, S_a, of radius a, with boundary $\Gamma_{0,a} \cup \Gamma_{1,a} \cup \tilde{\Gamma}_a$, where $\Gamma_{l,a}, l = 0, 1$ are the two straight sides of S_a and $\tilde{\Gamma}_a$ is the curved side of S. We consider the problem

$$-\Delta u = f \text{ in } S, \tag{1a}$$

$$u(x, 0) = g_0(x),\ x > 0 \tag{1b}$$

$$u(r\cos\omega, r\sin\omega) = g_1(r),\ r > 0. \tag{1c}$$

We assume that the functions f and g_l are sufficiently smooth and that $g_0(0) = g_1(0)$. We also suppose that

$$u \in H^1(S), \quad u(x, y) \equiv 0 \text{ for } r = \sqrt{x^2 + y^2} \geq 1. \tag{2}$$

From Lemma II.1;1 we know that $\hat{u} = \mathcal{K}u$ satisfies

$$\begin{cases} -\hat{u}_{\theta\theta}(\zeta, \theta) + \zeta^2 \hat{u}(\zeta, \theta) = \hat{f}(\zeta - 2i, \theta),\ \eta < 0, \\ \hat{u}(\zeta, 0) = \hat{g}_0(\zeta),\ \hat{u}(\zeta, \omega) = \hat{g}_1(\zeta), \quad \eta < 0. \end{cases} \tag{3}$$

From the development in §I.1, in particular the Paley-Wiener theorem, we have

$$\hat{u}(\zeta, \theta) \text{ is defined and holomorphic for } \eta < 0,$$

$$\hat{f}(\zeta, \theta) \text{ is defined and holomorphic for } \eta < -1,$$

$$\hat{g}_l(\zeta) \text{ is defined and holomorphic for } \eta < 0,\ l = 0, 1.$$

Furthermore, each of these functions is square integrable on any line $\Im\zeta \equiv$ const in the appropriate half plane, as well as the functions $\zeta\hat{u}$ and $\hat{u}_\theta$.

The two point boundary value problem (3) has constant coefficients and can be solved by the method of "variation of parameters". It is readily seen that the problem has a unique solution for any right hand sides, $\hat{f}$, $\hat{g}_0$, and $\hat{g}_1$, if and only if $\zeta \neq \pm\pi ni/\omega$ for n a non-zero integer. We may explicitly write the solution. Let a be a real number and set

$$k(\zeta, a) = \frac{e^{a\zeta} - e^{-a\zeta}}{2\zeta} = \zeta^{-1} \sinh a\zeta.$$

We note that for fixed a, k is an entire function with simple zeros at $\zeta = i\pi n/a$, $n = \pm 1, \pm 2, \cdots$. Also k is even in ζ. Also, for fixed η, $|k(\xi + i\eta, a)| \sim (1/2|\xi|)\exp(|a||\xi|)$, as $|\xi| \to \infty$. Using this function we have

$$\begin{aligned}\hat{u}(\zeta,\theta) = &\int_0^\theta \hat{f}(\zeta - 2i, \varphi)\frac{k(\zeta,\omega-\theta)k(\zeta,\varphi)}{k(\zeta,\omega)}d\varphi \\ &+ \int_\theta^\omega \hat{f}(\zeta - 2i, \varphi)\frac{k(\zeta,\theta)k(\zeta,\omega-\varphi)}{k(\zeta,\omega)}d\varphi \\ &+ \hat{g}_0(\zeta)\frac{k(\zeta,\omega-\theta)}{k(\zeta,\omega)} + \hat{g}_1(\zeta)\frac{k(\zeta,\theta)}{k(\zeta,\omega)}.\end{aligned} \tag{4}$$

We next take the inverse transform of the right side of (4) to obtain a formula for u. From the asymptotic behavior of $k(\zeta, a)$ we find that for large $|\xi|$

$$\frac{k(\zeta,\omega-\theta)k(\zeta,\varphi)}{k(\zeta,\omega)} \sim \frac{1}{2|\xi|}e^{-|\xi|(\theta-\varphi)}, \quad 0 < \varphi < \theta,$$

$$\frac{k(\zeta,\omega-\varphi)k(\zeta,\theta)}{k(\zeta,\omega)} \sim \frac{1}{2|\xi|}e^{-|\xi|(\varphi-\theta)}, \quad \theta < \varphi < \omega,$$

$$\frac{k(\zeta,\omega-\theta)}{k(\zeta,\omega)} \sim e^{-|\xi|\theta},$$

$$\frac{k(\zeta,\theta)}{k(\zeta,\omega)} \sim e^{-|\xi|(\omega-\theta)}.$$

Let us write, for convenience, $\hat{u} = \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4$, the 4 terms denoting the 4 integrals on the right side of (4). Suppose f, g_0, and g_1 are smooth functions. Then f is bounded in S, so f^* is bounded on $[0,\infty) \times [0,\omega]$. Similarly, g_l^* is bounded on $[0,\infty)$. Hence the integrals defining $\hat{f}(\zeta,\theta)$ and $\hat{g}_l(\zeta)$ are absolutely convergent for $\eta < 0$. Hence for $|\xi| \geq 1$ and $\eta < 2$, we get, for fixed ζ and θ,

$$\begin{aligned}|\hat{u}_1(\zeta,\theta)| &= \frac{1}{\sqrt{2\pi}}\left|\int_{\varphi=0}^\theta \int_{\sigma=0}^\infty f^*(\sigma,\varphi)\frac{k(\zeta,\omega-\theta)k(\zeta,\varphi)}{k(\zeta,\omega)}e^{-i(\zeta-2i)\sigma}d\sigma d\varphi\right| \\ &\leq C\int_0^\infty e^{-2\sigma}d\sigma \\ &< \infty.\end{aligned}$$

In the same way, we find that the integral defining $\hat{u}_2$ is absolutely convergent for fixed ζ and θ and $|\hat{u}_2(\zeta,\theta)| \leq C$. For large ξ we can get sharper estimates for $\hat{u}_1$ and $\hat{u}_2$. Assuming that $|\xi| > 1$ and $\eta < 2$ we use the asymptotic behavior of k to obtain

$$\begin{aligned}|\hat{u}_1(\zeta,\theta)| &\leq \frac{C}{|\xi|}\int_{\varphi=0}^\theta \int_{\sigma=0}^\infty e^{-|\xi|(\theta-\varphi)}e^{(\eta-2)\sigma}d\sigma d\varphi \\ &= \frac{C}{(2-\eta)|\xi|}\int_{\varphi=0}^\theta e^{-|\xi|(\theta-\varphi)}d\varphi \\ &\leq \frac{C}{|\xi|^2}.\end{aligned}$$

A similar estimate holds for $\hat{u}_2$. We also obtain, for $|\xi| \geq 1$,

$$|\hat{u}_3(\zeta,\theta)| \leq Ce^{-|\xi|\theta}, \quad |\hat{u}_4(\zeta,\theta)| \leq Ce^{-|\xi|(\omega-\theta)}.$$

Using these estimates we take the inverse transform,

$$
\begin{aligned}
u^*(\tau,\theta) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\zeta,\theta) e^{i\zeta\tau} d\xi \\
=& \frac{1}{\sqrt{2\pi}} \int_0^{\theta} \int_{\xi=-\infty}^{\infty} \hat{f}(\zeta - 2i, \varphi) \frac{k(\zeta,\omega-\theta)k(\zeta,\varphi)}{k(\zeta,\omega)} e^{i\zeta\tau} d\xi d\varphi \\
&+ \frac{1}{\sqrt{2\pi}} \int_{\theta}^{\omega} \int_{\xi=-\infty}^{\infty} \hat{f}(\zeta - 2i, \varphi) \frac{k(\zeta,\theta)k(\zeta,\omega-\varphi)}{k(\zeta,\omega)} e^{i\zeta\tau} d\xi d\varphi \\
&+ \frac{1}{\sqrt{2\pi}} \int_{\xi=-\infty}^{\infty} \hat{g}_0(\zeta) \frac{k(\zeta,\omega-\theta)}{k(\zeta,\omega)} e^{i\zeta\tau} d\xi + \frac{1}{\sqrt{2\pi}} \int_{\xi=-\infty}^{\infty} \hat{g}_1(\zeta) \frac{k(\zeta,\theta)}{k(\zeta,\omega)} e^{i\zeta\tau} d\xi.
\end{aligned}
\tag{5}
$$

In this formula, $\zeta = \xi + i\eta$. Using the above estimates for $\hat{u}_l$, we find that the integrals in (5) are absolutely convergent for $\eta < 0$, $\eta \neq \eta_n$, and $0 < \theta < \omega$, where $\eta_n = \pi n/\omega$. We write $u^* = u_1^* + u_2^* + u_3^* + u_4^*$, corresponding to the 4 terms on the right side of (5).

The formula (5) for u^* has a curious feature. In the right side of (5), the number η defines the line along which the integrals are taken. The left side, on the other hand, provides a formula for the solution u of (1), and so is independent of η for $\eta < 0$, $\eta \neq \eta_n$. To explain how this can happen, we note that the integrands that define the functions u_l^* are holomorphic in a strip. Specifically, since $\hat{f}(\eta,\theta)$ is holomorphic in the half plane $\eta < -1$, the integrands in the formulas for u_l^*, $l = 0,1$, are holomorphic in the strip $-\pi/\omega < \eta < 1$. Since g_l, $l = 1,2$, is holomorphic in the half plane $\eta < 0$, the integrals in the formulas for u_l^*, $l = 3,4$, are holomorphic in the strip $-\pi/\omega < \eta < 0$. It can be shown, using the Cauchy integral theorem, that each of these integrals is independent of η in the appropriate range. Hence the left side of (5) is indeed independent of η in an appropriate interval.

We now seek to move the line of integration outside the strip of holomorphy of the integrand. For this, we must make some more assumptions on the data f and g_l. Suppose that

$$
f \in C^{k-1}(\bar{S}_1) \text{ and } f(0,0) = D^1 f(0,0) = \cdots = D^{k-2} f(0,0) = 0. \tag{6a}
$$

Then $|f(x,y)| \leq Cr^{k-1}$ in $\bar{S}$ so $|f^*(\tau,\theta)| \leq C\exp(-(k-1)\tau)$. Recalling that $f^*(\tau,\theta) = 0$ for $\tau < 0$, it follows that the integral defining $\hat{f}(\zeta,\theta)$ is absolutely convergent for $\eta < k-1$, $\hat{f}(\zeta,\theta)$ is holomorphic in this half plane, and $|\hat{f}(\zeta,\theta)| \leq C/(k-1-\eta)$ for $\eta < k-1$. Similarly, if

$$
g_l \in C^{k+1}([0,1]) \text{ and } g_l(0) = g_l'(0) = \cdots = g_l^{(k)}(0) = 0, \tag{6b}
$$

then $|g_l(r)| \leq Cr^{k+1}$ in $[0,\infty)$, so $|g_l^*(\tau)| \leq C\exp(-(k+1)\tau)$ on R^1, $\hat{g}_l(\zeta)$ is holomorphic in the half plane $\eta < k+1$, and $|\hat{g}_l(\zeta)| \leq C/(k+1-\eta)$ for $\eta < k+1$. The constant C depends on the derivatives of g_l of order $\leq k+1$ in $[0,1]$. Summarizing, we have shown that (6) implies that

$$
|\hat{f}(\zeta - 2i,\theta)| \leq \frac{C}{k+1-\eta}, \quad |\hat{g}_l(\zeta)| \leq \frac{C}{k+1-\eta}, \quad \text{for } \eta < k+1. \tag{7}
$$

The shift of two derivatives in the requirements (6a) and (6b) is necessitated by the argument $\zeta - 2i$ in (5). The constant C in (7) depends on $|D^{k-1}f(x,y)|$ and $|D^{k+1}g_l(r)|$ for $0 \leq r \leq 1$.

We now deform the line of integration in the integrals defining u^*. For $\Xi > 0$ let $u_{l,\Xi}^*(\tau,\theta,\eta)$ denote the integral that defines u_l^*, with the integration carried along the line $\zeta = \xi + i\eta$ from $\xi = -\Xi$ to $\xi = \Xi$. (See the figure.) This function is defined for $\eta < k+1$ and for $\eta \neq \eta_n$. Let $\eta' \in (-\pi/\omega, 0)$ and let $\eta'' > 0$ with $\eta'' \neq \eta_n$. Then

$$
u_{l,\Xi}^*(\tau,\theta,\eta') = u_{l,\Xi}^*(\tau,\theta,\eta'') + v_l^*(\tau,\theta,\eta'') + R_{l,\Xi}^*(\tau,\theta,\eta',\eta'') + R_{l,-\Xi}^*(\tau,\theta,\eta',\eta''),
$$

where $R^*_{l,\pm\Xi}$ denotes the integral along the vertical sides $\xi = \pm\Xi$ from $\eta = \eta'$ to $\eta = \eta''$, and v^*_l denotes the sum of the residues of the integrand in the rectangle. Since the poles of the integrand are at $\zeta = \eta_n i$, v^*_l is independent of Ξ. Suppose $0 < \theta < \omega$. Using (7) and the estimates for k given above, we find that $R^*_{l,\pm\Xi} \to 0$ as $\Xi \to \infty$. Hence $\lim u^*_{l,\Xi}(\tau,\theta,\eta'')$ exists. Denoting this limit by $w^*_l(\tau,\theta,\eta'')$ and setting $v^* = \sum v^*_l$, $w^* = \sum w^*_l$, we obtain

$$u^*(\tau,\theta) = v^*(\tau,\theta,\eta) + w^*(\tau,\theta,\eta), \quad 0 < \eta < k+1, \quad \eta \neq \eta_n. \tag{8}$$

As will be seen, (8) provides an expansion of u^* into a sum of singular functions, v^*, and a remainder w^* that is smoother than u^*. We now study the functions v^* and w^*.

The function v^* depends on the poles ζ_n with $\eta_n < \eta$. If $\eta < \pi/\omega$, there are no poles in the strip $0 < \Im\zeta < \eta$, and $v^*(\tau,\theta,\eta) = 0$. If $N = [\omega\eta/\pi] > 0$ and $\omega\eta/\pi \neq$ integer, there are N poles in the strip. A computation shows that

$$k(\zeta,\omega) \sim (-1)^n \frac{\omega^2}{n\pi i}(\zeta - \eta_n i) \text{ as } \zeta \to \zeta_n = \eta_n i.$$

Hence the poles of each of the integrands is simple.We write $v^*_N(\tau,\theta)$ instead of $v^*(\tau,\theta,\eta)$, and we obtain

$$\begin{aligned} v^*_N(\tau,\theta) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} e^{-\pi n\tau/\omega} \{ &\sin\frac{\pi n(\omega-\theta)}{\omega} \int_{\varphi=0}^{\theta} \hat{f}((\frac{n\pi}{\omega}-2)i,\varphi)\sin\frac{n\pi\varphi}{\omega}d\varphi \\ &+ \sin\frac{\pi n\theta}{\omega}\int_{\varphi=\theta}^{\omega}\hat{f}((\frac{\pi n}{\omega}-2)i,\varphi)\sin\frac{\pi n(\omega-\varphi)}{\omega}d\varphi\} \\ &+ \frac{\sqrt{2\pi}}{\omega}\sum_{n=1}^{N}(-1)^{n+1}e^{-\pi n\tau/\omega}\{\hat{g}_0(\frac{\pi n}{\omega}i)\sin\frac{\pi n(\omega-\theta)}{\omega} + \hat{g}_1(\frac{\pi n}{\omega}i)\sin\frac{n\pi\theta}{\omega}\}. \end{aligned}$$

Since $\sin \pi n(\omega-\theta)/\omega = (-1)^{n+1}\sin \pi n\theta/\omega$, we obtain

$$\begin{aligned} v^*_N(\tau,\theta) = \sqrt{\frac{2}{\pi}}\sum_{n=1}^{N}\frac{1}{n}e^{-\pi n\tau/\omega}\{&\sin\frac{\pi n\theta}{\omega}\int_{\varphi=0}^{\omega}\hat{f}((\frac{\pi n}{\omega}-2)i,\varphi)\sin\frac{\pi n\varphi}{\omega}d\varphi \\ &+ \frac{\sqrt{2\pi}}{\omega}\sum_{n=1}^{N}e^{-\pi n\tau/\omega}\sin\frac{\pi n\theta}{\omega}\{\hat{g}_0(\frac{\pi n}{\omega}i) + (-1)^{n+1}\hat{g}_1(\frac{\pi n}{\omega}i)\}. \end{aligned}$$

To study this expression, it is convenient to define some linear functionals. We set

$$\Lambda'_n(g) = \hat{g}(\frac{\pi n}{\omega}i),$$

$$\Lambda''_n(f) = \frac{1}{n}\int_0^{\omega}\hat{f}((\frac{\pi n}{\omega}-2)i,\varphi)\sin\frac{\pi n\varphi}{\omega}d\varphi.$$

Then we have

$$v^*_N(\tau,\theta) = \sum_{n=1}^{N} e^{-\pi n\tau/\omega}\sin\frac{\pi n\theta}{\omega}\{\sqrt{\frac{2}{\pi}}\Lambda''_n(f) + \frac{\sqrt{2\pi}}{\omega}\Lambda'_n(g_0) + (-1)^{n+1}\frac{\sqrt{2\pi}}{\omega}\Lambda'_n(g_1)\}. \tag{9}$$

If the linear functionals are written in terms of the original independent variables, one obtains

$$\Lambda'_n(g) = \frac{1}{\sqrt{2\pi}}\int_0^1 r^{-(1+\pi n/\omega)}g(r)dr, \tag{10a}$$

$$\Lambda''_n(f) = \frac{1}{\sqrt{2\pi}n}\int\int r^{-\pi n/\omega}\sin\frac{\pi n\theta}{\omega}f(x,y)dxdy. \tag{10b}$$

We want to verify that the linear functionals $\Lambda'_n(g_l)$ and $\Lambda''_n(f)$ are finite. Since g_l satisfies (6b), $|g_l(r)| \leq Cr^{k+1}$. Since $\eta < k+1$ and $\pi n/\omega < \eta$, it follows that $k - \pi n/\omega > -1$, so the integral defining $\Lambda'_n(g_l)$ is absolutely convergent. A similar argument shows that the integral defining $\Lambda''_n(f)$ is absolutely convergent. This discussion is summarized in the following lemma.

Lemma 1. *If f and g_l satisfy (6), then the function v_N is given by*

$$v_N(x,y) = \sum_{n=1}^{N} r^{\pi n/\omega} \sin\frac{\pi n\theta}{\omega} \left\{ \sqrt{\frac{2}{\pi}}\Lambda_n''(f) + \frac{\sqrt{2\pi}}{\omega}\Lambda_n'(g_0) + (-1)^{n+1}\frac{\sqrt{2\pi}}{\omega}\Lambda_n'(g_1) \right\}. \tag{11}$$

where the linear functionals are given by (10a,b). The integrals defining these linear functionals are absolutely convergent.

The function $w^*(\tau,\theta,\eta)$ is given by the right side of (5) with the line of integration taken on $\Im\zeta = \eta > 0$. Therefore $w^*(\tau,\theta,\eta) = \mathcal{F}^{-1}\hat{w}(\zeta,\theta)$, where $\hat{w}(\zeta,\theta)$ is given by (4) or, equivalently, $\hat{w}(\zeta,\theta)$ is the solution of the problem (3) with $\eta > 0$. We use (4) to estimate $\hat{w}$ and its derivatives. The result is contained in the following lemma.

Lemma 2. *The function $w(x,y,\eta)$ depends only on N instead of η. Suppose f and g satisfy (6), and further, suppose that for some $m \le k-1$, one has $g_l^{(j)}(1) = 0$ for $1 \le j \le m+1$ and $D^j f(x,y) = 0$ on $r = 1$ for $1 \le j \le m-1$. If $m = 0$ we place no restriction on f. Then $w_N \in C^m(\bar{S})$ and*

$$|D^m w_N(x,y)| \le \frac{C}{k+1-\eta} r^{\eta-m} \text{ for } \eta_N < \eta < k+1. \tag{12}$$

Proof. Referring to (8), since v^* depends on N rather than η, and u^* is independent of η, w^* depends on N rather than η. To prove the inequality we first obtain bounds for $\hat{g}_l$ and $\hat{f}$ that improve (7). Let $n = m+2$, so $n \le k+1$. From (6b) we have $|g_l^{(n)}(r)| \le Cr^{k+1-n}$. Since $D_\tau^n g^*(\tau)$ is a linear combination of $r^j D_r^j g_l(r)$ for $1 \le j \le n$, it follows that

$$|D_\tau^n g_l^*(\tau)| \le Ce^{-(k+1)\tau}, \quad n \le k+1. \tag{13}$$

Since $\exp(-i\zeta\tau) = (-i\zeta)^{-n} D_\tau^n \exp(-i\zeta\tau)$,

$$\begin{aligned} \hat{g}_l(\zeta) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty g^*(\tau) e^{-i\zeta\tau} d\tau \\ &= \frac{(-1)^n}{\sqrt{2\pi}(i\zeta)^n} \int_0^\infty g^*(\tau) D_\tau^n e^{-i\zeta\tau} d\tau \quad , \\ &= \frac{1}{\sqrt{2\pi}(i\zeta)^n} \int_0^\infty [D_\tau^n g^*(\tau)] e^{-i\zeta\tau} d\tau \end{aligned}$$

where we have used in the integrations by parts the fact that $g_l^*(\tau) = \cdots = D_\tau^{n-1} g^*(\tau) = 0$ at $\tau = 0, \infty$. Using (13) we obtain

$$|\hat{g}_l(\zeta)| \le \frac{C}{(k+1-\eta)[|\zeta|^{m+2}+1]}, \text{ for } \eta < k+1. \tag{14}$$

In a similar way we obtain

$$|\hat{f}(\zeta - 2i,\theta)| \le \frac{C}{(k+1-\eta)[|\zeta|^{m}+1]}, \text{ for } \eta < k+1. \tag{15}$$

Recall that the integrals on the right side of (4), when evaluated at $\Im\zeta = \eta$, give the functions $\hat{w}_l$. We thus obtain from (15)

$$|\hat{w}_l(\zeta,\theta)| \le \frac{C}{(k+1-\eta)[|\zeta|^{m+2}+1]}, \quad \eta < k+1, \quad l = 1,2,$$

and similarly from (14),

$$|\hat{w}_l(\zeta,\theta)| \le \frac{C}{(k+1-\eta)[|\zeta|^{m+2}+1]}, \quad \eta < k+1, \quad l = 3,4.$$

It follows that

$$|D_\tau^m w^*(\tau,\theta,\eta)| \le \frac{C}{k+1-\eta} e^{-\eta\tau}, \quad \eta < k+1.$$

Since $D_r^m w_N$ is a linear combination of $e^{m\tau} D_\tau^j w_N^*$ for $1 \le j \le m$, we obtain $|D_r^m w_N(x,y)| \le C/(k+1-\eta)r^{\eta-m}$. A similar bound is readily obtained for $r^{-m} D_\theta^m w_N(x,y)$. ∎

We next consider what to do if f and g_l do not satisfiy the hypotheses (6a,b). For this, it is convenient to assume that the angle ω is not a rational multiple of π; i.e., ω/π is not a rational number. The following lemma explains the situation.

Lemma 3. *Let f, g_0, and g_1 be smooth functions. Suppose $\omega/\pi \ne$ integer. Then there is a linear polynomial $P(x,y)$ such that, setting $G_0(x) = g_0(x) - P(x,0)$, and $G_1(r) = g_1(r) - P(r\cos\omega, r\sin\omega)$, then $G_l(0) = G_l'(0) = 0, l = 1,2$. Furthermore let $k \ge 2$ and suppose ω/π is not a rational number of the form p/q with $q \le k$. Then there is a polynomial $P(x,y)$ of degree $\le k$ such that, setting $F = f + \Delta P$ and letting G_0 and G_1 be as above,*

(16a) $$F(0,0) = D^1F(0,0) = \cdots = D^{k-2}F(0,0) = 0.$$

(16b) $$G_l(0) = G_l'(0) = \cdots = G_l^{(k)}(0) = 0,$$

Proof. It is easily seen that Δ maps polynomials of degree k *onto* polynomials of degree $k-2$. Let $q(x,y)$ be the Taylor polynomial of f of degree $\le k-2$. Then there is a polynomial $Q(x,y)$ of degree $\le k$ such that $-\Delta Q = q$. Setting $F = f + \Delta Q = f - q$, we see that F satisfies (16a). Let $p_0(r)$ be the Taylor polynomial of $g_0(r) - Q(r,0)$ of degree $\le k$, and let $p_1(r)$ be the Taylor polynomial of $g_1(r) - Q(r\cos\omega, r\sin\omega)$ of degree $\le k$. Let us write $p_0(r) = \sum_0^k a_j r^j, p_1(r) = \sum_0^k b_j r^j$. Since $g_0(0) = g_1(0)$, it follows that $a_0 = b_0$. Let $P_l(x,y)$, $l = 0,1$, be the harmonic polynomials defined, in terms of polar coordinates, by

$$P_0(x,y) = a_0 + \sum_1^k \frac{a_j}{\sin j\omega} r^j \sin j(\omega - \theta),$$

$$P_1(x,y) = \sum_1^k \frac{b_j}{\sin j\omega} r^j \sin j\theta.$$

Since ω is not a rational multiple of π, the denominators do not vanish. Then $P(x,y) = Q(x,y) + P_0(x,y) + P_1(x,y)$ is the desired polynomial. We have $\Delta P = \Delta Q$, so $F = f + \Delta Q$ satisfies (16a). Also $G_0(r) = g_0(r) - P(r,0) = g_0(r) - Q(r,0) - \sum_0^k a_j r^j$ satisfies (16b), and similarly for $G_1(r)$. ∎

Assembling the above lemmas, we obtain the following theorem.

Theorem 1. *Let k be a non-negative integer. If $k > 0$, suppose $f \in C^{k-1}(\bar{S})$, $g_l \in C^{k+1}([0,\infty))$, and suppose ω/π is not a rational number of the form p/q with $q \le k$. If $k = 0$, suppose $f \in C^0(\bar{S})$, $g_l \in C^1([0,\infty))$, and suppose $\omega \ne \pi$ or 2π. Let $u \in H^1(S)$ be a weak solution of (1) with $u \equiv 0$ for $r \ge 1$. Let N be the largest integer such that $\pi N/\omega < k+1$. Then*

(17) $$u = v + w$$

where v is a function of the form (11), and $w \in C^k(\bar{S})$ and satisfies (12) with $\eta \in (\pi N/\omega, k+1)$.

Proof. Let $P(x,y)$ be the polynomial given by Lemma 3. Let $\psi \in C_0^\infty(R^2)$ with $\psi \equiv 1$ near the origin and $\psi \equiv 0$ for $x^2+y^2 \geq 1/2$. Let $u_1 = \psi(u-P)$, so $u = u_1 + (1-\psi)(u-P) + P$. Let $f_1 = -\Delta u_1$, $g_{10}(x) = u_1(x,0)$, $g_{11}(r) = u_1(r\cos\omega, r\sin\omega)$. These functions satisfy (6) and u_1 satisfies (2). Hence u_1 satisfies the hypotheses of Lemmas 1 and 2, so we may write $u_1 = v_{1,N} + w_{1,N}$, where $w_{1,N} \in C^k(S)$ satisfies (14). Since $(1-\psi)u \in C^k(S)$, we conclude that

$$u = v_{1,N} + w_{1,N} + (1-\psi)(u-P) + P, \tag{18}$$

where $v_{1,N}$ is of the form (11) and the other functions are in $C^k(S)$ and satisfy (14). ■

Notice that this theorem does not provide the full regularity that one expects from an elliptic operator; if f has $k-1$ derivatives and g has $k+1$ derivatives, w should have $k+1$ derivatives inside S. We will obtain this full regularity in a later section by paying closer attention to the properties of the boundary value problem (3).

The decomposition in Theorem 1 was obtained by moving the line of integration $\eta =$ const in (5) to larger values of η. One might ask what would result from moving the line of integration to *lower* values of η. If, for example, the line of integration were moved below $\eta = -\eta_1$, the residue of pole of the integrand at $\zeta = -\eta_1 i$ would give a term of the form $v = r^{-\pi/\omega}\sin(\pi\theta/\omega)$. This term becomes infinite at $r = 0$. On the other hand, the integral along the lower line gives a "remainder" term that is not smooth, and has a singularity at $r = 0$ that cancels the singularity of v. This deformation of the line of integration does not produce any information concerning the solution u.