## 1.1. Further remarks on the problem in a sector

The preceding section contains estimates for the solution of the Laplace equation in a sector, *under the assumption that the solution vanishes for* $r > 1$. This assumption was made to avoid the use of weighted Sobolev spaces. In this section we indicate briefly what can be expected without this vanishing assumption. We shall consider the problem

$$(1) \qquad -\Delta u = f \text{ in } S, \ u = 0 \text{ on } \Gamma.$$

Let $f \in C_0^\infty(S)$. Then $\hat{f}(\zeta,\theta)$ is analytic in the whole $\zeta$-plane. Let $\hat{u}(\zeta,\theta)$ be the solution to the two point boundary value problem

$$(2) \qquad \begin{cases} -\hat{u}_{\theta\theta}(\zeta,\theta) + \zeta^2\hat{u}(\zeta,\theta) = \hat{f}(\zeta - 2i,\theta), \\ \hat{u}(\zeta,0) = \hat{u}(\zeta,\omega) = 0. \end{cases}$$

Thus $\hat{u}$ is well-defined for $\zeta \neq j\alpha i$, $j = \pm 1, \pm 2, \cdots$, and is analytic in the $\zeta$-plane with these points removed. Furthermore, from Lemma II.1;2, $\hat{u}$ satisfies the estimate

$$(3) \qquad (1+\xi^2)\int_0^\omega |\hat{u}(\xi+i,\theta)|^2 d\theta + \int_0^\omega |\hat{u}_\theta(\xi+i,\theta)|^2 d\theta \leq C\int_0^\omega |\hat{f}(\xi-2i,\theta)|^2 d\theta.$$

The constant $C$ in (3) is independent of $\xi$. Since $\hat{u}$ vanishes at $\theta = 0$ and $\theta = \omega$ (3) implies

$$(4) \qquad \int_0^\omega |\hat{u}(\xi+i,\theta)|^2 d\theta \leq C\int_0^\omega |\hat{u}_\theta(\xi+i,\theta)|^2 d\theta \leq C'\int_0^\omega |\hat{f}(\xi-2i,\theta)|^2 d\theta.$$

Let $u(x) = \big(\mathcal{K}^{-1}\hat{u}(\cdot + i,\theta)\big)(x)$. Inequalities (3) and (4) imply

$$(5) \qquad \int\!\!\int_S [|\nabla u|^2 + r^{-2}u^2]dx \leq C\int\!\!\int_S r^2 f^2 dx.$$

We want to show that $u$ satisfies (1). For this let $v \in C_0^\infty(S)$. Setting $\zeta = \xi$ in (2), multiplying both sides of (2) by $\overline{\hat{v}(\xi,\theta)}$, integrating over $(\xi,\theta) \in \mathbb{R}\times(0,\omega)$, and integrating by parts, we obtain

$$\int_{-\infty}^\infty\int_0^\omega \hat{u}_\theta(\xi,\theta)\overline{\hat{v}_\theta(\xi,\theta)}d\theta d\xi + \int_{-\infty}^\infty\int_0^\omega \xi^2\hat{u}(\xi,\theta)\overline{\hat{v}(\xi,\theta)}d\theta d\xi = \int_{-\infty}^\infty\int_0^\omega \hat{f}(\xi-2i,\theta)\overline{\hat{v}(\xi,\theta)}d\theta d\xi.$$

Since $\int_{-\infty}^\infty \hat{f}(\xi-2i,\theta)\overline{\hat{v}(\xi,\theta)}d\xi = \int_{-\infty}^\infty \hat{f}(\xi-i,\theta)\overline{\hat{v}(\xi-i,\theta)}d\xi$, we obtain

$$\int_{-\infty}^\infty\int_0^\omega \hat{u}_\theta(\xi,\theta)\overline{\hat{v}_\theta(\xi,\theta)}d\theta d\xi + \int_{-\infty}^\infty\int_0^\omega \xi^2\hat{u}(\xi,\theta)\overline{\hat{v}(\xi,\theta)}d\theta d\xi = \int_{-\infty}^\infty\int_0^\omega \hat{f}(\xi-i,\theta)\overline{\hat{v}(\xi-i,\theta)}d\theta d\xi.$$

In the primal variables this equation becomes

$$(6) \qquad \int\!\!\int_S \nabla u\cdot\nabla v dx = \int\!\!\int_S f v dx \text{ for } v \in C_0^\infty(S).$$

Hence if $f \in C_0^\infty(S)$ there is a weak solution $u$ of (1) which satisfies (5). Since any $f \in L_2(S)$ can be approximated by functions $v \in C_0^\infty(S)$, we conclude that for any $f \in L_2(S)$ there is weak solution $u$ of (1) which satisfies (5).

The inequality (5) does not imply that $u \in H^1(S)$ since (5) does not imply that $u$ is square integrable "at infinity". (Inequality (5) does imply stronger integrability at the origin, however.) Higher order weighted inequalities may be obtained in the same way.

For a simpler establishment of the solvability of (1), we use with the Poincarè-type inequality

$$\int\int_S r^{-2}u^2 dx \leq C \int\int_S |\nabla u|^2 dx \text{ for } u \in H_0^1(S). \tag{7}$$

Let $f$ satisfy

$$\int\int_S r^2 f^2 dx < \infty. \tag{8}$$

Since

$$|\int\int_S fv dx| \leq \left(\int\int_S r^2 f^2 s dx\right)^{1/2} \left(\int\int_S r^{-2} v^2 s dx\right)^{1/2} \leq C \left(\int\int_S r^2 f^2 s dx\right)^{1/2} \|v\|_1,$$

the linear functional $v \mapsto \int\int_S fv dx$ is bounded on $H_0^1(S)$. Applying the Lax-Milgram lemma one concludes that if $f$ satisfies (8) there is a $u \in H_0^1(S)$ such that (6) holds. Furthermore, $u$ satisfies (5).