1. The equation $-\Delta u = f$ in a sector - some estimates

In this section the equation $\Delta u = f$ is transformed with $\mathcal{K}$, and some estimates for solutions of the transformed equation are obtained. These estimates are used in the following sections to obtain expansions of the solution in terms of singular functions of the problem. Three sets of boundary conditions are considered: Dirichlet conditions, Neumann conditions, and a mixed set of boundary conditions, one Dirichlet, one Neumann, on the two sides of S.

Let S be the sector of angle ω defined, in terms of polar coordinates, by $S = \{(x, y) : 0 < \theta < \omega\}$. We include the case $\omega = 2\pi$. Let $\Gamma = \Gamma_0 \cup \Gamma_1$ be the boundary of S, where Γ_0 and Γ_1 are the two sides of S. We shall also consider the truncated sector, S_a, of radius a, with boundary $\Gamma_{0,a} \cup \Gamma_{1,a} \cup \tilde{\Gamma}_a$, where $\Gamma_{l,a}, l = 0, 1$ are the two straight sides of S_a and $\tilde{\Gamma}_a$ is the curved side of S. Let $\alpha = \pi/\omega$.

We first consider the equation

$$-\Delta u = f \text{ in } S, \tag{1}$$

with Dirichlet boundary conditions

$$u(x, 0) = g_0(x),\ x > 0,\ u(r\cos\omega, r\sin\omega) = g_1(r),\ r > 0. \tag{2}$$

We will suppose that $f \in L_2(S)$, $g_l \in H^{3/2}(R_+)$, $l = 0, 1$, and $g_0(0) = g_1(0)$. This means that there is a function $G \in H^2(S)$ such that $G(x, 0) = g_0(x)$, $x > 0$, and $G(r\cos\omega, r\sin\omega) = g_1(r)$, $r > 0$. (See [Grisvard, 1].) We define $\mathcal{Y}_D^2(S)$ to be the set of data $\{f, g_0, g_1\}$ which satisfies these conditions. A function $u \in H^1(S)$ is called a *weak solution* of (1), (2), if $u = g_l$ on Γ_l $l = 0, 1$, and if

$$\int\int_S \nabla u \cdot \nabla\varphi dx = \int\int_S f\varphi dx,\ \varphi \in H_0^1(S), \tag{3}$$

where by $f(\varphi)$ is meant the value of the linear functional f on φ.

We start with a formal transformation of the problem. In the $\tau\theta$ variables we have, from (1), (2),

$$\begin{cases} -u^*_{\tau\tau}(\tau, \theta) - u^*_{\theta\theta}(\tau, \theta) = e^{-2\tau} f^*(\tau, \theta), \\ u^*(\tau, 0) = g_0^*(\tau),\ u^*(\tau, \omega) = g_1^*(\tau). \end{cases} \tag{4}$$

Taking the Fourier transform of (4), i.e., the Kondrat'yev transform of (1), we obtain formally

$$\begin{cases} -\hat{u}_{\theta\theta}(\zeta, \theta) + \zeta^2 \hat{u}(\zeta, \theta) = \hat{f}(\zeta - 2i, \theta), \\ \hat{u}(\zeta, 0) = \hat{g}_0(\zeta),\ \hat{u}(\zeta, \omega) = \hat{g}_1(\zeta). \end{cases} \tag{5}$$

We must justify the derivation of (5). For this, and for later analysis, we assume that u satisfies

$$u(x, y) \equiv 0 \text{ for } r = \sqrt{x^2 + y^2} \geq 1. \tag{6}$$

Thus, $f(x) = 0$ for $r \geq 1$ and $g_l(r) = 0$, $l = 0, 1$, for $r \geq 1$.

Lemma 1. *Let $\{f, g_0, g_1\} \in \mathcal{Y}_D^2(S)$, let $u \in H^1(S)$ be a weak solution of (1), (2), and suppose u satisfies (6). Then $\hat{u} = \mathcal{K}u$ satisfies $\hat{u}(\zeta, 0) = \hat{g}_0(\zeta)$ and $\hat{u}(\zeta, \omega) = \hat{g}_1(\zeta)$ for $\eta < 0$ and, for any $\psi \in H_0^1(0, \omega)$,*

$$\int_0^\omega [\hat{u}_\theta(\zeta, \theta)\psi_\theta(\theta) + \zeta^2 \hat{u}(\zeta, \theta)\psi(\theta)] d\theta = \int_0^\omega \hat{f}(\zeta - 2i, \theta)\psi(\theta) d\theta,\ \eta < 0. \tag{7}$$

Proof. Since $u \in H^s(S_1) = \tilde{H}^s(S_1)$ for $0 < s < 1$, from Theorem I.4;1, $\hat{u} \in \mathcal{H}^s(0,\omega)$. Hence $\hat{u}(\zeta,\theta)$ is analytic in the half plane $\eta < 0$. Since $u \in H^s(S_1)$ for $s < 1$, $g_l \in H^\sigma(0,1)$ for $\sigma < \frac{1}{2}$. Hence $g_l \in H_0^\sigma(0,1)$, and from Theorem I.2;2, $g_l \in \mathcal{H}^\sigma$, so $\hat{g}_l$ is analytic in the half plane $\eta < 0$. From Theorem I.4;4, $\hat{f}(\zeta - 2i, \theta)$ is analytic in the half plane $\eta < 0$. Transforming (3) to the $\tau\theta$ plane, one obtains

$$\int_{-\infty}^{\infty}\int_0^{\omega} [u_\tau^* \varphi_\tau^* + u_\theta^* \varphi_\theta^*] d\theta d\tau = \int_{-\infty}^{\infty}\int_0^{\omega} f^* \varphi^* e^{-2\tau} d\theta d\tau, \ \varphi \in H_0^1(S).$$

Applying Parseval's inequality,

$$\int_{-\infty}^{\infty}\int_0^{\omega} [\zeta^2 \hat{u}(\zeta,\theta)\hat{\varphi}(\zeta,\theta) + \hat{u}_\theta(\zeta,\theta)\hat{\varphi}_\theta(\zeta,\theta)] d\theta d\xi$$
$$= \int_{-\infty}^{\infty}\int_0^{\omega} \hat{f}(\zeta - 2i, \theta)\hat{\varphi}(\zeta,\theta) d\theta d\xi, \ \hat{\varphi} \in \mathcal{H}^s(\omega).$$

Since $\hat{u}(\zeta,\theta)$ and $\hat{f}(\zeta - 2i,\theta)$ are analytic for $\eta < 0$, we obtain (7). ■

The boundary value problem (5) has a solution provided $\zeta \neq j\alpha i$, $j = \pm 1, \pm 2, \cdots$. (A formula for the solution is given in (II.2.2;3).) Therefore, if $\eta \neq j\alpha$, the problem (5) has a solution for any ζ with $\Im\zeta = \eta$. It must not be presumed that this solution is the transform of the solution u to (1), (2). As will be seen, the function $\hat{u}(\zeta,\theta)$ does *not*, in general, have an analytic extension to an arbitrary half plane $\Im\zeta < \eta$. This fact is emphasized in the formulation of the next lemma and theorem, in which bounds for the solution of (5) are given on a line $\Im\zeta =$ constant.

Lemma 2. *Let $\eta/\alpha \neq$ integer and let $s \geq 2$. There is a $C(s,\eta) > 0$ such that if $\hat{F} \in H^{s-2}(0,\omega)$, the solution $\hat{U}(\zeta,\theta)$ of the boundary value problem*

$$-\hat{U}_{\theta\theta}(\zeta,\theta) + \zeta^2\hat{U}(\zeta,\theta) = \hat{F}(\theta),$$
$$\hat{U}(\zeta,0) = \hat{G}_0, \ \hat{U}(\zeta,\omega) = \hat{G}_1,$$

satisfies, for $\zeta = \xi + \eta i$,

$$(8) \quad \begin{aligned} &(1+\xi^2)^s \int_0^{\omega} |\hat{U}(\zeta,\theta)|^2 d\theta + ||\hat{U}(\zeta,\cdot)||^2_{H^s(0,\omega)} \\ &\leq C(s,\eta)(1+\xi^2)^{s-2}\int_0^{\omega} |\hat{F}(\theta)|^2 d\theta + C(s,\eta)||\hat{F}||^2_{H^{s-2}(0,\omega)} \\ &\quad + C(s,\eta)(1+\xi^2)^{s-1/2}[|\hat{G}_0|^2 + |\hat{G}_1|^2]. \end{aligned}$$

Proof. Let $||\cdot||_s$ denote the norm in $H^s(0,\omega)$. Suppose first that $\hat{G}_0 = \hat{G}_1 = 0$ and assume that $\hat{F}$ is smooth. Multiplying both sides of the differential equation by $\overline{\hat{U}(\zeta,\theta)}$, integrating over $(0,\omega)$, and taking the real part, we obtain

$$\int_0^{\omega} (|\hat{U}_\theta(\zeta,\theta)|^2 + (\xi^2 - \eta^2)|\hat{U}(\zeta,\theta)|^2) d\theta = \Re \int_0^{\omega} \hat{F}(\theta)\overline{\hat{U}(\zeta,\theta)} d\theta.$$

Suppose that $|\xi| > 2|\eta|$. Then

$$(9) \quad \int_0^{\omega} (|\hat{U}_\theta(\zeta,\theta)|^2 + \tfrac{1}{2}\xi^2|\hat{U}(\zeta,\theta)|^2) d\theta \leq |\int_0^{\omega} \hat{F}(\theta)\overline{\hat{U}(\zeta,\theta)} d\theta|, \ |\xi| > |\eta|.$$

Bounding the right side by $||\hat{F}||_{-1}||\hat{U}(\zeta,\cdot)||_1$, we obtain

$$(10) \quad ||\hat{U}(\zeta,\cdot)||_1 \leq C||\hat{F}||_{-1}.$$

Applying Schwarz's inequality to the right side of (9), we get

$$|\xi|||\hat{U}_\theta(\zeta,\cdot)||_0+|\xi|^2||\hat{U}(\zeta,\cdot)||_0\le C||\hat{F}||_0. \tag{11}$$

Next, we use the differential equation, the triangle inequality, and (11) to obtain

$$||\hat{U}_{\theta\theta}(\zeta,\cdot)||_0\le C||\hat{F}||_0.$$

An inductive argument then shows that for any integer $k\ge 2$,

$$||D_\theta^k\hat{U}||_0\le C|\xi|^k||\hat{F}||_0+||D_\theta^{k-2}\hat{F}||_0. \tag{12}$$

These inequalities prove (8) if $s=$ integer ≥ 2 and $|\xi|>2|\eta|$. Now let $|\xi|\le 2|\eta|$. Since $\eta/\alpha\ne$ integer, the boundary value problem is solvable for all ξ. Upon differentiation of the solution formula we obtain (8) for $s=$ integer ≥ 2 and with $C(s,\eta)$ replaced by a constant $C'(s,\xi,\eta)$. The solution formula is a continuous function of ζ so the constants $C'(s,\xi,\eta)$ are bounded from above for $\xi\in[-2|\eta|,2|\eta|]$. Hence (8) holds for all $\xi\in\mathbb{R}$. and $s=$ integer ≥ 2. The inequality (8) for $2<s\ne$ integer then follows by interpolation.

Now take the case $\hat{F}=0$. The solution $\hat{U}$ is

$$\hat{U}(\zeta,\theta)=\hat{G}_0\frac{\sinh\zeta(\omega-\theta)}{\sinh\zeta\omega}+\hat{G}_1\frac{\sinh\zeta\theta}{\sinh\zeta\omega}.$$

We find that for large values of ξ,

$$|\hat{U}(\zeta,\theta)|\le C[|\hat{G}_0|e^{-|\xi|\theta}+|\hat{G}_1|e^{-|\xi|(\omega-\theta)}],$$

$$|D_\theta^k\hat{U}(\zeta,\theta)|\le C|\xi|^k[|\hat{G}_0|e^{-|\xi|\theta}+|\hat{G}_1|e^{-|\xi|(\omega-\theta)}].$$

The inequality (8) for $s=$ integer ≥ 1 follows from these bounds. The inequality (8) for $1<s\ne$ integer then follows by interpolation. ■

We obtain from Lemma 2 an inequality for the solution of the two point boundary value problem (5) under the assumption that the data has more regularity. Let $s>2$, and let $\mathcal{Y}_D^s(S)$ denote the set of triples $\{f,g_0,g_1\}$ such that $f\in H^{s-2}(S)$, $g_l\in H^{s-1/2}(R_+)$, $l=0,1$, and $g_0(0)=g_1(0)=0$. $\mathcal{Y}_D^s(S)$ is a Hilbert space with the Cartesian product norm.

Theorem 1. *Suppose $s>2$ satisfies $(s-1)/\alpha\ne$ integer. Then there is a constant $C(s)>0$ such that if $\{f,g_0,g_1\}\in\mathcal{Y}_D^s$, the solution $w(\xi,\theta;s-1)$ of the two point boundary value problem*

$$-w_{\theta\theta}(\xi,\theta;s-1)+(\xi+(s-1)i)^2w(\xi,\theta;s-1)=\hat{f}(\xi+(s-3)i,\theta),$$

$$w(\xi,0;s-1)=\hat{g}_0(\xi+(s-1)i),$$

$$w(\xi,\omega;s-1)=\hat{g}_1(\xi+(s-1)i),$$

then

$$\int_{-\infty}^{\infty}\int_0^\omega(1+\xi^2)^s|w((\xi,\theta;s-1)|^2d\theta d\xi+\int_{-\infty}^{\infty}||w(\xi,\cdot;s-1)||_{H^s(0,\omega)}d\xi$$
$$\le C(s)||\{f,g_0,g_1\}||^2_{\mathcal{Y}_D^s}.$$

Proof. The result follows from Lemma 2 by integrating both sides of (8) with respect to ξ. ■

We now consider Neumann boundary conditions; that is, we consider (1) with the boundary conditions

$$
\begin{aligned}
&-u_y(x,0) = h_0(x),\ x>0 \\
&-(\sin\omega)u_x(r\cos\omega, r\sin\omega) \\
&\quad +(\cos\omega)u_y(r\cos\omega, r\sin\omega) = h_1(r),\ r>0.
\end{aligned}
\tag{13}
$$

Note that if (1), (13) has a solution, the data $\{f, h_0, h_1\}$ must satisfy the compatibility condition

$$
\int\int_S f\,dx + \int_0^\infty [h_0(x) + h_1(x)]dx = 0. \tag{14}
$$

We suppose that $h_l \in H^{1/2}(R_+)$, $l = 0, 1$, and that $f \in L_2(S)$. We define $\mathcal{Y}_N^2(S)$ to be the set of data $\{f, h_0, h_1\}$ which satisfies these conditions. A function $u \in H^1(S)$ is defined to be a *weak solution* of (1), (13), if u satisfies

$$
\int\int_S \nabla u\cdot\nabla\varphi dx = \int_0^\infty [h_0(r)\varphi(r,0) + h_1(r)\varphi(r\cos\omega, r\sin\omega)]dr + \int\int_S f\varphi dx \text{ for } \varphi\in H^1(S). \tag{15}
$$

The transformed problem is now

$$
\begin{cases}
-\hat{u}_{\theta\theta}(\zeta,\theta) + \zeta^2\hat{u}(\zeta,\theta) = \hat{f}(\zeta-2i,\theta), \\
\hat{u}_\theta(\zeta,0) = \hat{h}_0(\zeta-i),\ \hat{u}_\theta(\zeta,\omega) = \hat{h}_1(\zeta-i).
\end{cases}
\tag{16}
$$

The compatibility condition (14) transforms to

$$
\int_0^\omega \hat{f}(-2i,\theta)d\theta + \hat{h}_0(-i) + \hat{h}_1(-i) = 0. \tag{17}
$$

The proof of the following lemma is the same as the proof of Lemma 1, and so is omitted.

Lemma 3. *Let $\{f, h_0, h_1\} \in \mathcal{Y}_N^2(S)$, let $u \in H^1(S)$ be a weak solution of (1), (13), and suppose u satisfies (6). Then for any $\psi \in H^1(0,\omega)$,*

$$
\int_0^\omega [\hat{u}_\theta(\zeta,\theta)\psi_\theta(\theta) + \zeta^2\hat{u}(\zeta,\theta)\psi(\theta)]d\theta = \int_0^\omega \hat{f}(\zeta-2i,\theta)\psi(\theta)d\theta + \hat{h}_0(\zeta-i)\psi(0) + \hat{h}_1(\zeta-i)\psi(\omega),\ \eta<0. \tag{18}
$$

The boundary value problem (16) has a solution provided $\zeta \neq j\alpha i$, $j = \pm1, \pm2, \cdots$. (A formula for the solution is given in (II.2.3;5).) Therefore, if $\eta \neq j\alpha$, the problem (16) has a solution for any ζ with $\Im\zeta = \eta$. It must not be presumed that this solution is the transform of the solution u to (1), (13). As will be seen, the function $\hat{u}(\zeta,\theta)$ does *not*, in general, have an analytic extension to an arbitrary half plane $\Im\zeta < \eta$. This fact is emphasized in the formulation of the next lemma and theorem, in which bounds for the solution of (16) are given on a given line $\Im\zeta =$ constant.

Lemma 4. *Let $\eta/\alpha \neq$ integer and let $s \geq 2$. There is a $C(s,\eta) \neq 0$ such that if $\hat{F} \in H^s(0,\omega)$, the solution $\hat{U}(\zeta,\theta)$ of the boundary value problem*

$$
\begin{aligned}
&-\hat{U}_{\theta\theta}(\zeta,\theta) + \zeta^2\hat{U}(\zeta,\theta) = \hat{F}(\theta), \\
&\hat{U}_\theta(\zeta,0) = \hat{H}_0,\ \hat{H}_\theta(\zeta,\omega) = \hat{H}_1,
\end{aligned}
$$

satisfies, for $\zeta = \xi + \eta i$,

$$
\begin{aligned}
&(1+\xi^2)^s \int_0^\omega |\hat{U}(\zeta,\theta)|^2 d\theta + ||\hat{U}(\zeta,\cdot)||^2_{H^s(0,\omega)} \\
&\qquad \leq C(s,\eta)(1+\xi^2)^{s-2}\int_0^\omega |\hat{F}(\theta)|^2 d\theta + C(s,\eta)||\hat{F}||^2_{H^{s-2}(0,\omega)} \\
&\qquad\quad + C(s,\eta)(1+\xi^2)^{s-3/2}[|\hat{H}_0|^2 + |\hat{H}_1|^2]
\end{aligned}
\tag{19}
$$

Proof. Suppose first that $\hat{H}_0 = \hat{H}_1 = 0$ and assume that $\hat{F}$ is smooth. In this case, the same argument as used in Lemma 2 yields (19). Now take the case $\hat{F} = 0$. The solution $\hat{U}$ is

$$\hat{U}(\zeta,\theta) = \hat{H}_0 \frac{\cosh \zeta(\omega-\theta)}{\zeta \sinh \zeta\omega} + \hat{H}_1 \frac{\cosh \zeta\theta}{\zeta \sinh \zeta\omega}.$$

We find that for large values of ξ,

$$|\hat{U}(\zeta,\theta)| \leq C|\zeta|^{-1}[|\hat{H}_0|e^{-|\xi|\theta} + |\hat{H}_1|e^{-|\xi|(\omega-\theta)}],$$

$$|D_\theta^k \hat{U}(\zeta,\theta)| \leq C|\xi|^{k-1}[|\hat{H}_0|e^{-|\xi|\theta} + |\hat{H}_1|e^{-|\xi|(\omega-\theta)}].$$

The inequality (19) for $s =$ integer ≥ 2 follows from these bounds. The inequality (19) for $2 < s \neq$ integer then follows by interpolation. ∎

We obtain from Lemma 4 an inequality for the solution of the two point boundary value problem (16) under the assumption that the data has more regularity. Let $s > 2$, and let $\mathcal{Y}_N^s(S)$ denote the set of triples $\{f, h_0, h_1\}$ such that $f \in H^{s-2}(S)$ and $h_l \in H^{s-3/2}(R_+)$, $l = 0, 1$. $\mathcal{Y}_N^s(S)$ is a Hilbert space with the Cartesian product norm.

Theorem 2. *Suppose $s > 2$ satisfies $(s-1)/\alpha \neq$ integer. Then there is a constant $C(s) > 0$ such that if $\{f, h_0, h_1\} \in \mathcal{Y}_N^s$, the solution $w(\xi,\theta; s-1)$ of the two point boundary value problem*

$$-w_{\theta\theta}(\xi,\theta;s-1) + (\xi + (s-1)i)^2 w(\xi,\theta;s-1) = \hat{f}(\xi + (s-3)i,\theta),$$

$$w_\theta(\xi,0;s-1) = \hat{h}_0(\xi + (s-2)i),$$

$$w_\theta(\xi,\omega;s-1) = \hat{h}_1(\xi + (s-2)i),$$

then

$$\int_{-\infty}^{\infty}\int_0^\omega (1+\xi^2)^s |w((\xi,\theta;s-1)|^2 d\theta d\xi + \int_{-\infty}^{\infty} ||w(\xi,\cdot;s-1)||_{H^s(0,\omega)} d\xi$$
$$\leq C(s)||\{f,h_0,h_1\}||^2_{\mathcal{Y}_N^s}.$$

Proof. The result follows from Lemma 4 by integrating both sides of (8) with respect to ξ. ∎

We now consider mixed boundary conditions; that is, we consider (1) with the boundary conditions

(20)
$$\begin{aligned} u(x,0) &= g_0(x),\ x > 0 \\ -(\sin\omega)u_x(r\cos\omega, r\sin\omega) & \\ +(\cos\omega)u_y(r\cos\omega, r\sin\omega) &= h_1(r),\ r > 0. \end{aligned}$$

We suppose that $g_0 \in H^{3/2}(R_+)$, $h_1 \in H^{1/2}(R_+)$, and that $f \in L_2(S)$. We define $\mathcal{Y}_M^2(S)$ to be the set of data $\{f, g_0, h_1\}$ which satisfies these conditions. A function $u \in H^1(S)$ is defined to be a *weak solution* of (1), (20), if $u = g_0$ on Γ_0 and if u satisfies

(21)
$$\int\int_S \nabla u \cdot \nabla\varphi dx = \int_0^\infty h_1(r)\varphi(r\cos\omega, r\sin\omega)dr + \int\int_S f\varphi dx \text{ for } \varphi \in H^1(S),\ \varphi = 0 \text{ on } \Gamma_0.$$

The transformed problem is now

(22)
$$\begin{cases} -\hat{u}_{\theta\theta}(\zeta,\theta) + \zeta^2\hat{u}(\zeta,\theta) = \hat{f}(\zeta - 2i,\theta), \\ \hat{u}(\zeta,0) = \hat{g}_0(\zeta),\ \hat{u}_\theta(\zeta,\omega) = \hat{h}_1(\zeta - i). \end{cases}$$

The proof of the following lemma is the same as the proof of Lemma 1, and so is omitted.

Lemma 5. *Let $\{f, g_0, h_1\} \in \mathcal{Y}^2_M(S)$, let $u \in H^1(S)$ be a weak solution of (1), (20), and suppose u satisfies (6). Then for any $\psi \in H^1(0,\omega)$ with $\psi(0) = 0$,*

$$\int_0^\omega [\hat{u}_\theta(\zeta,\theta)\psi_\theta(\theta) + \zeta^2 \hat{u}(\zeta,\theta)\psi(\theta)]d\theta = \int_0^\omega \hat{f}(\zeta - 2i, \theta)\psi(\theta)d\theta + \hat{h}_1(\zeta - i)\psi(\omega),\ \eta < 0. \tag{23}$$

The boundary value problem (22) has a solution provided $\zeta \neq (j - \frac{1}{2})\alpha i$, $j = \pm 1, \pm 2, \cdots$. (A formula for the solution is given in (II.2.4;3).) Therefore, if $2\eta/\alpha \neq$ odd integer, the problem (22) has a solution for any ζ with $\Im\zeta = \eta$. It must not be presumed that this solution is the transform of the solution u to (1), (20). As will be seen, the function $\hat{u}(\zeta,\theta)$ does *not*, in general, have an analytic extension to an arbitrary half plane $\Im\zeta < \eta$. This fact is emphasized in the formulation of the next lemma and theorem, in which bounds for the solution of (22) are given on a given line $\Im\zeta =$ constant.

Lemma 6. *Let $2\eta/\alpha \neq$ odd integer and let $s \geq 2$. There is a $C(s,\eta) \neq 0$ such that if $\hat{F} \in H^s(0,\omega)$, the solution $\hat{U}(\zeta,\theta)$ of the boundary value problem*

$$-\hat{U}_{\theta\theta}(\zeta,\theta) + \zeta^2\hat{U}(\zeta,\theta) = \hat{F}(\theta),$$
$$\hat{U}(\zeta,0) = \hat{G}_0,\ \hat{H}_\theta(\zeta,\omega) = \hat{H}_1,$$

satisfies, for $\zeta = \xi + \eta i$,

$$\begin{aligned}(1+\xi^2)^s \int_0^\omega |\hat{U}(\zeta,\theta)|^2 d\theta &+ ||\hat{U}(\zeta,\cdot)||^2_{H^s(0,\omega)} \\ &\leq C(s,\eta)(1+\xi^2)^{s-2}\int_0^\omega |\hat{F}(\theta)|^2 d\theta + C(s,\eta)||\hat{F}||^2_{H^{s-2}(0,\omega)} \\ &\quad + C(s,\eta)(1+\xi^2)^{s-1/2}|\hat{G}_0|^2 + C(s,\eta)(1+\xi^2)^{s-3/2}|\hat{H}_1|^2\end{aligned} \tag{24}$$

Proof. Suppose first that $\hat{G}_0 = \hat{H}_1 = 0$ and assume that $\hat{F}$ is smooth. In this case, the same argument as used in Lemma 2 yields (24). Now take the case $\hat{F} = 0$. The solution $\hat{U}$ is

$$\hat{U}(\zeta,\theta) = \hat{G}_0 \frac{\cosh\zeta(\omega-\theta)}{\cosh\zeta\omega} + \hat{H}_1 \frac{\sinh\zeta\theta}{\zeta\cosh\zeta\omega}.$$

We find that for large values of ξ,

$$|\hat{U}(\zeta,\theta)| \leq C[|\hat{G}_0|e^{-|\xi|\theta} + |\zeta|^{-1}|\hat{H}_1|e^{-|\xi|(\omega-\theta)}],$$

$$|D^k_\theta \hat{U}(\zeta,\theta)| \leq C|\xi|^k|\hat{G}_0|e^{-|\xi|\theta}C|\xi|^{k-1}|\hat{H}_1|e^{-|\xi|(\omega-\theta)}.$$

The inequality (24) for $s =$ integer ≥ 2 follows from these bounds. The inequality (24) for $2 < s \neq$ integer then follows by interpolation. ■

We obtain from Lemma 6 an inequality for the solution of the two point boundary value problem (22) under the assumption that the data has more regularity. Let $s > 2$, and let $\mathcal{Y}^s_M(S)$ denote the set of triples $\{f, h_0, h_1\}$ such that $f \in H^{s-2}(S)$, $g_0 \in H^{s-1/2}(R_+)$, and $h_1 \in H^{s-3/2}(R_+)$. $\mathcal{Y}^s_M(S)$ is a Hilbert space with the Cartesian product norm.

Theorem 3. *Suppose $s > 2$ satisfies $2(s-1)/\alpha \neq$ odd integer. Then there is a constant $C(s) > 0$ such that if $\{f, g_0, h_1\} \in \mathcal{Y}^s_M$, the solution $w(\xi,\theta; s-1)$ of the two point boundary value problem*

$$-w_{\theta\theta}(\xi,\theta;s-1) + (\xi + (s-1)i)^2 w(\xi,\theta;s-1) = \hat{f}(\xi + (s-3)i, \theta),$$

$$w(\xi, 0; s-1) = \hat{g}_0(\xi + (s-1)i),$$
$$w_\theta(\xi, \omega; s-1) = \hat{h}_1(\xi + (s-2)i),$$

then

$$\int_{-\infty}^{\infty} \int_0^{\omega} (1+\xi^2)^s |w((\xi, \theta; s-1)|^2 d\theta d\xi + \int_{-\infty}^{\infty} ||w(\xi, \cdot; s-1)||_{H^s(0,\omega)} d\xi$$
$$\leq C(s) ||\{f, g_0, h_1\}||^2_{\mathcal{Y}^s_M}.$$

Proof. The result follows from Lemma 6 by integrating both sides of (24) with respect to ξ. ■

We conclude the section with an estimate for the solution of (1) in either the Dirichlet, Neumann, or mixed case, *under the assumption that the solution is in* $H^2(S)$. This estimate will be of use in later sections. In the statement of the result, $\gamma_l u$ denotes the value, or trace, of u on the side Γ_l of S, $l = 0, 1$, and $\gamma_l u_n$ denotes the value, or trace, of the normal directional derivative of u on the side Γ_l.

Lemma 7. *There is a constant $C > 0$ such that if $u \in H^2(S)$, one has the inequalities*

$$||u||_{H^2(S)} \leq C||\Delta u||_{H^0(S)} + C \sum_{l=0,1} ||\gamma_l u||_{H^{3/2}(\Gamma_l)}, \tag{25}$$

$$||u||_{H^2(S)} \leq C||\Delta u||_{H^0(S)} + C \sum_{l=0,1} ||\gamma_l u_n||_{H^{1/2}(\Gamma_l)}, \tag{26}$$

$$||u||_{H^2(S)} \leq C||\Delta u||_{H^0(S)} + C||\gamma_0 u||_{H^{3/2}(\Gamma_0)} + ||\gamma_1 u_n||_{H^{1/2}(\Gamma_1)}. \tag{27}$$

Proof. We use the identity

$$\begin{aligned} \int\int_S (\Delta u)^2 dx &= \int\int_S (u^2_{x_1x_1} + 2u_{x_1x_1}u_{x_2x_2} + u^2_{x_2x_2}) dx \\ &= \int\int_S (u^2_{x_1x_1} + 2u^2_{x_1x_2} + u^2_{x_2x_2}) dx \\ &\quad + 2\int_{\partial S} (u_{x_1x_1}u_{x_2}n_2 - u_{x_1x_2}u_{x_2}n_1) ds, \end{aligned} \tag{28}$$

valid for a function $u \in C^3(\bar{S})$ which vanishes outside a compact set. Suppose $\gamma_0 u = 0$ or $\gamma_0 u_n = \gamma_0 u_y = 0$. Then the integral in (28) that is taken along Γ_0 vanishes. Similarly, if $\gamma_1 u = 0$ or $\gamma_1 u_n = 0$, the integral along Γ_1 vanishes. Hence (25), (26), or (27) are established if $u \in C^3(\bar{S})$ vanishes outside a compact set and if, respectively, $\gamma_0 u = \gamma_1 u = 0$, or $\gamma_0 u_n = \gamma_1 u_n = 0$, or $\gamma_0 u = \gamma_1 u_n = 0$. By a limiting argument, (25), (26), or (27) are true if $u \in H^2(S)$ and if, respectively, $\gamma_0 u = \gamma_1 u = 0$, or $\gamma_0 u_n = \gamma_1 u_n = 0$, or if $\gamma_0 u = \gamma_1 u_n = 0$. There remains the problem of proving the inequalities when u does not satisfy the respective 0 boundary conditions. To prove (25) for any $u \in H^2(S)$, let $\mathcal{T}$ denote the closed subspace of $H^{3/2}(R_+) \times H^{3/2}(R_+)$ defined by the condition $\{g_0, g_1\} \in \mathcal{T}$ if $g_0(0) = g_1(0)$. Let $E : \mathcal{T} \to H^2(S)$ denote the extension operator, defined in [***]. Let $U = E\{\gamma_0 u, \gamma_1 u\}$. Thus, $||\Delta U||_{H^2(S)} \leq C_1 \sum ||\gamma_l u||_{H^{3/2}(\Gamma_l)}$. Let $v = u - U$. Since $\gamma_0 v = \gamma_1 v = 0$, we get, using (25) applied to v and the triangle inequality,

$$\begin{aligned} ||u||_{H^2(S)} \leq& ||v||_{H^2(S)} + ||U||_{H^2(S)} \\ \leq& C||\Delta v||_{H^0(S)} + C_1 \sum_{l=0,1} ||\gamma_l u||_{H^{3/2}(\Gamma_l)} \\ \leq& C||\Delta u||_{H^0(S)} + C_1 \sum_{l=0,1} ||\gamma_l u||_{H^{3/2}(\Gamma_l)}. \end{aligned}$$

A similar argument can be used to prove (26) and (27) for any $u \in H^2(S)$. ■

One knows that the solution of the Neumann problem (1), (13), is unique up to an additive constant. This is not contradicted by (26), since there is at most one solution that belongs to $H^2(S)$.