

3.1 Equivalent norms in the case $\omega = 2\pi$

In the case of a slit sector, $\omega = 2\pi$, Corollary I.3;2 identifies the fractional Sobolev space s and provides a formula for the norm on this space, namely $|||\hat{u}|||_s$. It is of interest to give a formula for the norm on $\tilde{H}^s(S_1)$ that does not require the Kondrat'yev transform of u . If $\omega < 2\pi$, such a formula is given in terms of the seminorm $|u|_s$. Since Lemma I.3;3 and Lemma I.3;7 require that $\omega < 2\pi$, we cannot use this seminorm for the slit sector, and indeed we do not expect it to be valid in the case of the slit sector.

In this section we define a new seminorm on $H^s(S)$. The new seminorm, which is smaller than $|u|_{s,S}$, is equivalent to $|u|_{s,S}$ if $\omega < 2\pi$, and in the case $\omega = 2\pi$ the new seminorm gives the correct norm for the interpolation spaces $[H^1(S), L_2(S)]_s$.

To describe our new seminorm we need some notation. If $x \in R^2$, set $x = (r \cos \theta, r \sin \theta)$, so $r = |x|$. We write $\theta = \arg(x)$ with $0 \leq \arg(x) < 2\pi$. Thus, $S = \{x : 0 < \arg(x) < \omega\}$. Let $(\alpha \in (0, 2\pi))$ be given. For $x \in S$ we let

$$\begin{aligned} \mathcal{W}_\alpha(x) &= \{y \in S : |\arg(x) - \arg(y)| < \alpha\} \\ &= \{y : \max\{0, \arg(x) - \alpha\} < \arg(y) < \min\{\omega, \arg(x) + \alpha\}\}. \end{aligned}$$

We then define the new seminorm, $N_s^{(\alpha)}(u)$, by

$$N_s^{(\alpha)}(u)^2 = \int_S \left\{ \int_{y \in \mathcal{W}_\alpha(x)} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dy \right\} dx.$$

In addition to the quantity $N_s^{(\alpha)}(u)$ we define new seminorms corresponding to the other seminorms defined in §I.3:

$$\begin{aligned} N_{s,1}^{(\alpha)}(u)^2 &= \int_S \left\{ \int_{\{t:(x_1+t, x_2) \in \mathcal{W}_\alpha(x)\}} \frac{|u(x_1+t, x_2) - u(x)|^2}{t^{1+2s}} dt \right\} dx, \\ N_{s,2}^{(\alpha)}(u)^2 &= \int_S \left\{ \int_{\{t:(x_1, x_2+t) \in \mathcal{W}_\alpha(x)\}} \frac{|u(x_1+t, x_2) - u(x)|^2}{t^{1+2s}} dt \right\} dx, \\ N_{s,\theta}^{(\alpha)}(u)^2 &= \int_0^\infty \int_0^\omega \left\{ \int_{\max(0, \theta-\alpha)}^{\min(\omega, \theta+\alpha)} \frac{|u^\#(r, \theta) - u^\#(r, \varphi)|^2}{|\theta - \varphi|^{1+2s}} d\varphi \right\} r^{1-2s} d\theta dr. \end{aligned}$$

Since the pairs of points that occur in the double integrals defining these norms form a subset of $S \times S$, we have

$$(1) \quad N_s^{(\alpha)}(u) \leq |u|_s, \quad N_{s,j}^{(\alpha)}(u) \leq |u|_{s,j}, \quad N_{s,\theta}^{(\alpha)}(u) \leq |u|_{s,\theta}.$$

If $\omega < \alpha$, then $y \in \mathcal{W}_\alpha$ for any $(x, y) \in S \times S$, so

$$(2) \quad N_s^{(\alpha)}(u) = |u|_s, \quad N_{s,j}^{(\alpha)}(u) = |u|_{s,j}, \quad N_{s,\theta}^{(\alpha)}(u) = |u|_{s,\theta} \text{ for } \omega \leq \alpha.$$

Our first result is

Lemma 1. *Let S be a sector with $\omega \leq 2\pi$. Then we have the seminorm equivalence*

$$c(s)N_{s,\theta}^{(\alpha)}(u)^2 \leq |u|_{s,\theta}^2 \leq C(s)N_{s,\theta}^{(\alpha)}(u)^2.$$

Proof. From (1) we may take $c(s) = 1$. From (I.3;22),

$$|u|_{s,\theta} \leq C \sum_j |u|_{s,\theta,S_j},$$

where $S_j \subset S$ is a sector with angle $< \alpha$. It is obvious that $|u|_{s,\theta,S_j} = N_{s,\theta,S_j}^{(\alpha)}(u) \leq N_{s,\theta,S}^{(\alpha)}(u)$, so $|u|_{s,\theta} \leq C(s)N_{s,\theta}^{(\alpha)}(u)$. Hence $|u|_{s,\theta} \sim N_{s,\theta}^{(\alpha)}(u)$ and the result follows. ■

Our next lemma establishes the seminorm equivalence of Lemma I.3;7 for the new seminorms.

Lemma 2. *Let S be a sector with angle $\omega \leq 2\pi$. Then there are constants $c(s, \alpha)$ and $C(s, \alpha)$ such that*

$$(3) \quad c(s, \alpha)[N_{s,1}^{(\alpha)}(u)^2 + N_{s,2}^{(\alpha)}(u)^2] \leq |u|_{s,r}^2 + N_{s,\theta}^{(\alpha)}(u)^2 \leq C(s, \alpha)[N_{s,1}^{(\alpha)}(u)^2 + N_{s,2}^{(\alpha)}(u)^2],$$

$$(4) \quad c(s, \alpha)[N_{s,1}^{(\alpha)}(u)^2 + N_{s,2}^{(\alpha)}(u)^2] \leq N_s^{(\alpha)}(u)^2 \leq C(s, \alpha)[N_{s,1}^{(\alpha)}(u)^2 + N_{s,2}^{(\alpha)}(u)^2],$$

Proof. Let $S_j = \{x : (j-1)\omega/n < \arg(x) < j\omega/n\}$. The sets S_1, \dots, S_n form a partition of S . Writing $\int_s = \sum_j \int_{S_j}$ in the definition of $N_{s,1}^{(\alpha/3)}(u)^2$, we obtain a corresponding sum $N_{s,1}^{(\alpha/3)}(u)^2 = \sum_1^n A_j$. The points x and $(x_1 + t, x_2)$ in the double integral that defines A_j satisfy

$$\frac{(j-1)\omega}{n} < \arg(x) < \frac{j\omega}{n}, \quad \frac{(j-1)\omega}{n} - \frac{\alpha}{3} < \arg((x_1 + t, x_2)) < \frac{j\omega}{n} + \frac{\alpha}{3}.$$

Thus, these points lie in a sector \tilde{S}_j of angle $\omega/n + 2\alpha/3$. Hence $A_j \leq |u|_{s,1,\tilde{S}_j}^2$. Pick n large enough so that $\omega/n < \alpha/3$. Then the angle of \tilde{S}_j is $< \alpha < \omega$. Hence we may apply Lemma I.3;7 to obtain $A_j \leq C_1(s, \alpha)[|u|_{s,r,\tilde{S}_j}^2 + |u|_{s,\theta,\tilde{S}_j}^2]$. Again, since the angle of \tilde{S}_j is $< \alpha$, $|u|_{s,\theta,\tilde{S}_j} \leq N_{s,\theta}^{(\alpha)}(u)$. Summing the resulting inequality over j , we obtain

$$N_{s,1}^{(\alpha)}(u)^2 \leq C[|u|_{s,r}^2 + N_{s,\theta}^{(\alpha)}(u)^2].$$

A similar argument bounds $N_{s,2}^{(\alpha)}(u)^2$, and the left inequality in (3) is proved. The inequality on the right side of (3) is proved in the same way, writing the integral $\int_s = \sum_j \int_{S_j}$ in the definition of $|u|_{s,r}^2$ or $N_{s,\theta}^{(\alpha/3)}(u)^2$. A similar argument proves (4). ■

We now state our norm equivalence.

Theorem 1. *Let S be a sector with $0 < \omega \leq 2\pi$. If $0 < \alpha < 2\pi$ and $0 < s < 1$, the quantity*

$$[N_s^{(\alpha)}(u)^2 + \|u\|_0^2]^{1/2}$$

provides an equivalent norm on $\tilde{H}^s(S_1)$.

Proof. The fact that $N_s^{(\alpha)}(u)$ provides another equivalent norm on $\tilde{H}^s(S_1)$ follows from Lemmas 1, 2, and I.3;8. ■