2. Mellin transforms of Sobolev spaces on $\mathbb{R}_+$

We are concerned with determining the Mellin transform of some Sobolev spaces. We start with the spaces $H^s(\mathbb{R}_+)$. We recall from [Lions-Magenes, 1968, §10.3] that if $0 < s < 1$, $||u||^2_{H^s(\mathbb{R}_+)} = ||u||^2_{L_2(\mathbb{R}_+)} + |u|^2_s$, where

$$|u|^2_s = \int_0^\infty \int_0^\infty \frac{|u(x+y) - u(x)|^2}{y^{1+2s}} dxdy.$$

Our first concern is with the seminorm $|u|_s$. To transform $|u|_s$ we require a lemma, which will also be used later.

Lemma 1. *Let $0 < \nu < 2$ and let $\zeta = \xi + \eta i$ be complex with $2\eta < \nu$. Let*

$$k(\zeta, \nu) = \int_0^\infty \frac{|e^{-i\zeta\gamma} - 1|^2}{[e^\gamma - 1]^\nu [1 - e^{-\gamma}]} d\gamma.$$

Then the integral is convergent, and k satisfies the inequality

$$c(\eta,\nu)(|\xi|^\nu + 1) \le k(\zeta,\nu) \le C(\eta,\nu)(|\xi|^\nu + 1),$$

where $c(\eta,\nu)$ and $C(\eta,\nu)$ are positive constants which depend only on η and ν.

Proof. If $\eta \le 0$ the integrand behaves like $e^{-\nu\gamma}$ near $\gamma = \infty$, whereas if $\eta > 0$ the integrand behaves like $e^{-(\nu - 2\eta)\gamma}$ near $\gamma = \infty$. The integrand behaves like $\gamma^{1-\nu}$ near $\gamma = 0$. Hence the integral that defines k is finite under the stated conditions. Using the identity $|e^{-ai+b} - 1|^2 = (e^b - 1)^2 + 4e^b \sin^2(a/2)$, we obtain $k = k_1 + k_2$ where

$$k_1(\eta,\nu) = \int_0^\infty \frac{[e^{\eta\gamma} - 1]^2}{[e^\gamma - 1]^\nu [1 - e^{-\gamma}]} d\gamma,$$

$$k_2(\zeta,\nu) = \int_0^\infty \frac{4e^{\eta\gamma}\sin^2(\xi\gamma/2)}{[e^\gamma - 1]^\nu [1 - e^{-\gamma}]} d\gamma.$$

Since the integral defining k_1 is convergence for η and ν in the allowed range, $k_1(\eta,\nu)$ is a finite positive number. To bound $k_2(\eta,\nu)$ we use the inequalities

$$c_1 \frac{\gamma}{\gamma+1} \le 1 - e^{-\gamma} \le C_1 \frac{\gamma}{\gamma+1},$$
$$c_1 \frac{\gamma e^\gamma}{\gamma+1} \le e^\gamma - 1 \le C_1 \frac{\gamma e^\gamma}{\gamma+1},$$

which are valid for $\gamma \ge 0$ and with $c_1 = e^{-1}$, $C_1 = 2$. We then obtain

$$4C_1^{-\nu-1} B(\zeta,\nu) \le k_2(\zeta,\nu) \le 4c_1^{-\nu-1} B(\zeta,\nu),$$

where

$$B(\zeta,\nu) = \int_0^\infty \frac{(\gamma+1)^{\nu+1}}{\gamma^{\nu+1}} e^{-(\nu-\eta)\gamma} \sin^2 \frac{\xi\gamma}{2} d\gamma.$$

From the hypotheses, $\nu - \eta > 0$. Hence $(\gamma+1)^{\nu+1} e^{-(\nu-\eta)\gamma} \le C$ and we obtain

$$B(\zeta,\nu) \le C \int_0^\infty \frac{\sin^2 \frac{\xi\gamma}{2}}{\gamma^{\nu+1}} d\gamma = C|\xi|^\nu \int_0^\infty \sigma^{-\nu-1} \sin^2(\sigma/2) d\sigma.$$

On the other hand,

$$\begin{aligned} B(\zeta,\nu) &\geq \int_0^1 \gamma^{-1-n} e^{-(\nu-\eta)\gamma} \sin^2 \frac{\xi\gamma}{2} d\gamma \\ &\geq e^{-(\nu-\eta)} \int_0^1 \gamma^{-1-n} \sin^2 \frac{\xi\gamma}{2} d\gamma \\ &= e^{-(\nu-\eta)} |\xi|^\nu \int_0^{|\xi|} \sigma^{-1-\nu} \sin^2 \tfrac{1}{2}\sigma d\sigma \\ &\geq \begin{cases} c|\xi|^\nu, & |\xi| \geq 1, \\ c|\xi|^2, & |\xi| \leq 1. \end{cases} \end{aligned}$$

The result follows from these inequalities. ■

We now give a formula for the Mellin transform of the semi-norm $|u|_s$. In fact, we shall consider the quantity

$$A_{s,a}(u) = \int_0^\infty \int_0^\infty \frac{|u(x+y) - u(x)|^2}{y^{1+2s}} x^a dx dy,$$

which, in the case $a = 0$, equals $|u|_s$.

Lemma 2. *Let $0 < s < 1$, $1 + a > 0$. Let u be a function on $\mathbb{R}_+$ which satisfies $\int_0^\infty x^{a-2s} u(x)^2 dx < \infty$. Then the Mellin transform $\hat{u}(\xi + \eta i)$ is defined and square integrable on the line $\eta = s - \frac{1}{2} - \frac{1}{2}a$. Furthermore, $\xi^s \hat{u}(\xi + (s - \frac{1}{2} - \frac{1}{2}a)i)$ is square integrable on $\mathbb{R}$ if and only if $A_{s,a}(u) < \infty$, and in this event, there are positive numbers $c(s,a)$ and $C(a,s)$ such that*

$$c(s,a) A_{s,a}(u) \leq \int_{-\infty}^\infty \left[|\xi|^{2s} + 1\right] |\hat{u}(\xi + (s - \tfrac{1}{2} - \tfrac{1}{2}a)i)|^2 d\xi \leq C(s,a) A_{s,a}(u). \tag{1}$$

Finally,

$$c(s,a) A_{s,a}(u) \leq ||e^{(s-\frac{1}{2}-\frac{1}{2}a)\tau} u^*||^2_{s,\mathbb{R}} \leq C(s,a) A_{s,a}(u). \tag{1a}$$

Proof. The existence and square integrability of the Mellin transform on the line $\eta = s - \frac{1}{2} - \frac{1}{2}a$ follows from the Parseval formula and the formula

$$\int_0^\infty x^{a-2s} u(x)^2 dx = \int_{-\infty}^\infty e^{(2s-a-1)\tau} u^*(\tau)^2 d\tau \int_{-\infty}^\infty \left(e^{(s-\frac{1}{2}-\frac{1}{2}a)\tau} u^*(\tau)\right)^2 d\tau.$$

Set $x = \exp(-\alpha)$, $y = \exp(-\beta)$. We then obtain

$$A_{s,a}(u) = \int_{-\infty}^\infty \int_{-\infty}^\infty |u(e^{-\alpha} + e^{-\beta}) - u(e^{-\alpha})|^2 e^{2s\beta - (1+a)\alpha} d\beta d\alpha.$$

Let $\gamma = \ln(1 + e^{\alpha-\beta})$. For fixed α, γ is a decreasing function of β with $\lim_{\beta\to-\infty} \gamma = \infty$, $\lim_{\beta\to\infty} \gamma = \ln 1 = 0$. Hence the map $(\alpha,\beta) \to (\alpha,\gamma)$ is (1-1) and sends $\mathbb{R}^2$ onto $\mathbb{R} \times \mathbb{R}_+$ with $\lim_{\beta\to-\infty} \gamma = \infty$, $\lim_{\beta\to\infty} \gamma = \ln 1 = 0$. Also, from the definition of γ,

$$e^{-\alpha} + e^{-\beta} = e^{\gamma-\alpha},\ e^\beta = e^\alpha/(e^\gamma - 1),\ \partial\gamma/\partial\beta = -[1 - e^{-\gamma}].$$

Since $u(e^{-\alpha}) = u^*(\alpha)$ and $u(e^{-\alpha} + e^{-\beta}) = u^*(\alpha - \gamma)$, we obtain

$$A_{s,a}(u) = \int_{\alpha=-\infty}^\infty \int_{\gamma=0}^\infty |u^*(\alpha-\gamma) - u^*(\alpha)|^2 \frac{e^{(2s-1-a)\alpha}}{[e^\gamma - 1]^{2s}[1 - e^{-\gamma}]} d\gamma d\alpha.$$

Since $e^{(s-1/2-a/2)\alpha}u^*(\alpha) \in L_2(\mathbb{R})$,

$$\mathcal{F}(e^{(s-1/2-a/2)\alpha}[u^*(\alpha-\gamma)-u^*(\alpha)])(\xi) = [e^{-i\gamma[\xi+(s-1/2-a/2)i]}-1]\hat{u}(\xi+(s-\tfrac{1}{2}-\tfrac{1}{2}a)i),$$

and we may use Parseval's formula to obtain

$$A_{s,a}(u) = \int_{-\infty}^{\infty} |\hat{u}(\xi+(s-\tfrac{1}{2}-\tfrac{1}{2}a)i)|^2 k(\xi+(s-\tfrac{1}{2}-\tfrac{1}{2}a)i, 0, 2s)d\xi.$$

Here, k is the function given in Lemma 1. Applying Lemma 1 with $\eta = s-\frac{1}{2}-\frac{1}{2}a$, $\nu = 2s$, we obtain the first result. The converse statement follows in a similar way, using Parseval's formula. To prove (1a), let $z^*(\tau) = \big(\mathcal{F}^{-1}\hat{u}(\cdot+(s-\frac{1}{2}-\frac{1}{2}a)i))\big)(\tau)$. Then $z^* \in H^s(\mathbb{R})$ and, using (1), the quantity $||z^*||^2_{s,\mathbb{R}}$ is equivalent to $A_{s,a}(u)$. Since $\big(\mathcal{F}e^{(s-\frac{1}{2}-\frac{1}{2}a)\tau}u^*\big)(\xi) = \hat{u}(\xi+(s-\frac{1}{2}-\frac{1}{2}a)i)$, one has $z^*(\tau) = e^{(s-\frac{1}{2}-\frac{1}{2}a)\tau}u^*(\tau)$, and (1a) follows. ■

For $0 < s < 1$ we define the subspace $H_0^s(\mathbb{R}_+)$ by interpolation:

$$H_0^s(\mathbb{R}_+) = [H_0^1(\mathbb{R}_+), L_2(\mathbb{R}_+)]_{1-s}.$$

We recall from Lions-Magenes [1968, Chapter 1, Theorem 11.1] that if $0 < s < \frac{1}{2}$, $H_0^s(\mathbb{R}_+) = H^s(\mathbb{R}_+)$, and if $\frac{1}{2} < s < 1$, $H_0^s(\mathbb{R}_+)$ is the closed subspace of $H^s(\mathbb{R}_+)$ consisting of functions which vanish at 0 [ibid, Chapter 1, Theorem, 11.5]. Thus, the relation

$$||u||^2_{H^s(\mathbb{R}_+)} \sim |u|^2_{s,\mathbb{R}_+} + ||u||^2_{L_2(\mathbb{R}_+)}, \quad s \neq \tfrac{1}{2}, \tag{2}$$

provides an equivalent norm in $H_0^s(\mathbb{R}_+)$. We also know from the same source that if $u \in H_0^s(\mathbb{R}_+)$,

$$\int_0^\infty \frac{1}{x^{2s}}|u(x)|^2 dx \leq C||u||^2_{H^s(\mathbb{R}_+)}, \ 0 < s < 1, \ s \neq \tfrac{1}{2}. \tag{3}$$

The inequality (3) is not true at $s = \frac{1}{2}$, and an extra weighting is required at $x = 0$ to describe the norm of $H_0^{1/2}(\mathbb{R}_+)$. In fact, we have [ibid, Chapter 1, Theorem 11.7]

$$||u||^2_{H_0^{1/2}(\mathbb{R}_+)} \sim |u|^2_{1/2,\mathbb{R}_+} + \int_0^\infty \frac{1}{x}|u(x)|^2 dx. \tag{4}$$

In any event we have

$$||u||^2_{H_0^s(\mathbb{R}_+)} \sim |u|^2_{s,\mathbb{R}_+} + ||u||^2_{L_2(\mathbb{R}_+)}, \quad 0 < s < 1. \tag{5}$$

Note that when $s = \frac{1}{2}$ our notation differs from customary usage. What we denote by $H_0^{1/2}(\mathbb{R}_+)$ is called $H_{00}^{1/2}(\mathbb{R}_+)$ in Lions-Magenes [1968].

These results conform with the Sobolev imbedding theorem: if $s > \frac{1}{2}$, functions in $H^s(\mathbb{R}_+)$ are continuous on $\mathbb{R}_+$, whereas if $s < \frac{1}{2}$, there are functions in $H^s(\mathbb{R}_+)$ which are not continuous. (A step function provides an example.) The transition exponent, $s = \frac{1}{2}$, is a special case, and in that case the interpolated norm is given by (4).

The following theorem identifies the Mellin transform of functions in $H_0^s(\mathbb{R}_+)$.

Theorem 1. *Let $0 < s \leq 1$. The Mellin transform of $H_0^s(\mathbb{R}_+)$ consists of all functions $\hat{u}(\zeta)$ which are analytic in the strip $-\frac{1}{2} < \eta < s - \frac{1}{2}$ and which are square integrable with respect to ξ on each line $\eta = \text{const} \in [-\frac{1}{2}, s - \frac{1}{2}]$. The quantity $X(\hat{u})$ defined by the formula*

$$X_s(\hat{u})^2 \equiv \int_{-\infty}^{\infty} |\hat{u}(\xi - \tfrac{1}{2}i)|^2 d\xi + \int_{-\infty}^{\infty} [\xi^2 + 1]^s |\hat{u}(\xi + (s - \tfrac{1}{2})i)|^2 d\xi < \infty. \tag{6}$$

provides an equivalent norm on $H_0^s(\mathbb{R}_+)$.

Proof. Let $u \in H_0^s(\mathbb{R}_+)$. Since $u \in L_2(\mathbb{R}_+)$, $e^{-\tau/2}u^* \in L_2(\mathbb{R})$. Since $\int x^{-2s}|u|^2 dx < \infty$ (in the case $s = 1/2$ this follows from (4)), $e^{(s-1/2)\tau}u^* \in L_2(\mathbb{R})$. Hence $\hat{u}(\zeta)$ is well defined and analytic in the strip $-1/2 < \eta < s - 1/2$. Using Lemma 2, we obtain $X_s(u) \leq C||u||_{H_0^s(\mathbb{R}_+)}$. Conversely, suppose $\hat{u}$ is analytic in the strip $-1/2 < \eta < s - 1/2$, and satisfies the hypotheses of the theorem. Then from the Paley-Wiener theorem, the integral that defines $u^* = \mathcal{F}^{-1}\hat{u}$ is absolutely convergent, u^* is independent of $\eta \in (-1/2, s - 1/2)$, and $||u||_{H_0^s(\mathbb{R}_+)} \leq CX_s(u) < \infty$. ■

It is in certain respects more convenient to work with Sobolev spaces on $I = (0, 1)$ instead of on $\mathbb{R}_+$. For $0 < s < 1$ we define the space $H_0^s(I)$ by interpolation:

$$H_0^s(I) = [H_0^1(I), L_2(I)]_{1-s}. \tag{7}$$

(In the case $s = \frac{1}{2}$, this is again not the customary usage.) Again, if $0 < s < \frac{1}{2}$, $H_0^s(I) = H^s(I)$, and if $\frac{1}{2} < s < 1$, $H_0^s(I)$ is the closed subspace of $H^s(I)$ consisting of functions which vanish at $x = 0$ and $x = 1$. Since functions in $H_0^s(I)$ vanish both at $x = 0$ and at $x = 1$, weights may be included at both these points to describe the norm of $H_0^s(I)$, in the case $s \neq \frac{1}{2}$, and these weights *must* be provided to describe the norm of $H_0^{1/2}(I)$. That is, if $u \in H_0^s(I)$,

$$\int_0^1 \frac{1}{x^{2s}(1-s)^{2s}} |u(x)|^2 dx \leq C||u||_{H^s(I)}, s \neq \tfrac{1}{2}.$$

On the other hand, this inequality is not true for $s = \frac{1}{2}$, but we have the norm equivalence

$$||u||^2_{H_0^{1/2}(I)} \sim |u|^2_{1/2,I} + \int_0^1 \frac{1}{x(1-x)} |u(x)|^2 dx. \tag{8}$$

The functions in $H_0^s(I)$ have a convenient extension property. If $u \in H_0^s(I)$, and if we extend u by 0 for $x > 1$ (we shall also call the extended function u), then $u \in H_0^s(\mathbb{R}_+)$ and $||u||_{s,\mathbb{R}_+} \leq C||u||_{s,I}$. In fact, for $0 < s < 1$,

$$||u||^2_{s,I} \sim |u|^2_{s,\mathbb{R}_+} + \int_0^1 \frac{1}{x^{2s}} |u(x)|^2 dx. \tag{9}$$

The right side of (9) gives the norm in $H_0^s(I)$ that is of most convenience for us.

Comparing (8) and (9), we see that by making the extension, we are able to drop the weight in the integral at $x = 1$. This is expressed by the inequality that if $u(x) \equiv 0$ for $x > 1$, then

$$\int_0^1 \frac{1}{1-x} |u(x)|^2 dx \leq C \left[\int_0^1 |u(x)|^2 dx + \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^2}{|x-y|^2} dx dy \right].$$

If $u \in H_0^s(I)$, by $\mathcal{M}u$ we mean $\mathcal{M}$ applied to the extended function. Thus,

$$(\mathcal{M}u)(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^1 x^{i\zeta - 1} u(x) dx.$$

We now determine the image $\mathcal{M}H_0^s(I)$. For this we define a space $\mathcal{H}^s$ of analytic functions as follows. Let $s \in \mathbb{R}$ and let $\mathcal{H}^s$ consists of the functions $\hat{u}(\zeta)$ which are analytic in the half space $\eta < s - \frac{1}{2}$, and for which

$$\sup_{\eta < s - \frac{1}{2}} \int_{-\infty}^{\infty} [1 + \xi^2]^s |\hat{u}(\xi + \eta i)|^2 d\xi < \infty. \tag{10}$$

We make $\mathcal{H}^s$ into a Hilbert space, defining a norm $|||\hat{u}|||_s$ by

$$|||\hat{u}|||_s^2 = \int_{-\infty}^{\infty} [1 + \xi^2]^s |\hat{u}(\xi + (s - \tfrac{1}{2})i)|^2)|^2 d\xi.$$

The spaces $\mathcal{H}^s$ form a decreasing family: if $s < t$, then $\mathcal{H}^s \supset \mathcal{H}^t$. We have

Theorem 2. *Let $0 \leq s \leq 1$. Then $\mathcal{M}H_0^s(I) = \mathcal{H}^s$ and $\mathcal{M} : H_0^s(I) \to \mathcal{H}^s$ is an invertible map. The quantity $|||\hat{u}|||_s$ provides an equivalent norm on $H_0^s(I)$. If $s < t$, then $\mathcal{H}^t$ is a dense subset of $\mathcal{H}^s$.*

Proof. Let $u \in H_0^s(I)$. Since $\int_0^\infty x^{-2s}|u|^2 dx < \infty$, $e^{(s-1/2)\tau}u^* \in L_2(\mathbb{R})$. Since $u(x) = 0$ for $x > 1$, $u^*(\tau) = 0$ for $\tau < 0$. From the Paley Wiener theorem, $\hat{u}(\xi + \eta i) = \mathcal{F}(e^{(s-1/2)\tau}u^*(\tau))(\zeta)$ is analytic for $\eta < s - \frac{1}{2}$ and

$$\int_{-\infty}^{\infty} |\hat{u}(\xi + \eta i)|^2 d\xi \leq \int_0^{\infty} x^{-2s} |u|^2 dx \text{ for } \eta \leq s - \tfrac{1}{2}.$$

Also, using Lemma 2,

$$\sup_{\eta \leq s - 1/2} \int_{-\infty}^{\infty} [1 + \xi^2]^s |\hat{u}(\xi + \eta i)|^2 d\xi \leq C(s) ||u||_{s,I}^2.$$

Conversely, suppose $\hat{u} \in \mathcal{H}^s$. From the Paley Wiener theorem, $e^{(s-1/2)\tau}u^* \in L_2(\mathbb{R})$ and $u^*(\tau) = 0$ for $\tau < 0$. Hence $u = \mathcal{M}^{-1}\hat{u}$ vanishes for $x > 1$ and $\int_0^1 x^{-2s}u^2 dx \leq C|||\hat{u}|||_s^2$. Using Lemma 2 and (9) we find that $||u||_{H_0^s(I)} \leq C|||\hat{u}|||_s$. ∎

As a consequence of Theorem 2 we obtain

Theorem 3. *The spaces $\mathcal{H}^s$, $0 \leq s \leq 1$ form an interpolating family:*

$$[\mathcal{H}^1, \mathcal{H}^0]_{1-s} = \mathcal{H}^s, \text{ for } 0 < s < 1. \tag{11}$$

Proof. Since $\mathcal{M} : H_0^s(I) \to \mathcal{H}^s$ is an isomorphism for $0 \leq s \leq 1$, the result follows from (7). ∎

As a corollary to this result, we obtain an interpolation result, that will be of use in §I.4, for another family of spaces. Let $\mathcal{H}^{s,\sigma}$ consist of the functions $\hat{u}(\zeta)$ which are analytic in the half-space $\eta < s + \sigma$ and for which

$$\sup_{\eta \leq s + \sigma} \int_{-\infty}^{\infty} [1 + \xi^2]^s |\hat{u}(\xi + \eta i)|^2 d\xi < \infty.$$

The norm in $\mathcal{H}^{s,\sigma}$ is defined by

$$|||\hat{u}|||_{s,\sigma}^2 = \int_{-\infty}^{\infty} [1 + \xi^2]^s |\hat{u}(\xi + (s + \sigma)i)|^2 d\xi.$$

We have

Corollary. *The spaces* $\mathcal{H}^{s,\sigma}$, $0 \le s \le 1$, *form an interpolating family:*

$$\mathcal{H}^{s,\sigma} = [\mathcal{H}^{1,\sigma}, \mathcal{H}^{0,\sigma}]_{1-s} \text{ for } 0 < s < 1.$$

Proof. From the definition, $\mathcal{H}^{s} = \mathcal{H}^{s,-1/2}$. The map T defined by

$$(T\hat{u})(\zeta) = \hat{u}(\zeta - (t + \tfrac{1}{2})i)$$

defines an isomorphism $T\mathcal{H}^{s} \to \mathcal{H}^{s,\sigma}$. The result then follows from Theorem 3. ■

Our next concern is with the spaces $H^{-s}(I)$. Recall that $H^{-1}(I)$ is defined to be the set of distributions on $\mathbb{R}$ which are bounded linear functionals on $H_0^1(I)$. Of course, the Riesz representation theorem identifies the bounded linear functionals on $H_0^1(I)$ with $H_0^1(I)$ itself. That is, if $v \in H_0^1(I)$ the map $u \to \int u'\bar{v}'dx$ is a bounded linear functional on $H_0^1(I)$, and any bounded linear functional on $H_0^1(I)$ can be obtained in this way. Writing $\int u'\bar{v}'dx = -\langle u, v''\rangle$, we see that the linear functional associated with v is also given by the distribution $f = -v''$. The norm of the distribution f is

$$||f||_{H^{-1}(I)} = \sup\left\{\frac{\langle f, v\rangle}{||v||_1} : v \in H_0^1(I)\right\}.$$

In particular, if $f \in L_2(I)$, then $||f||_{H^{-1}(I)} < \infty$ and $H^{-1}(I)$ is the closure of $L_2(I)$ under this norm.

We wish to give the Mellin transform of this construction. Let $\hat{f} \in \mathcal{H}^{-s}$, and let $\Lambda_{\hat{f}}$ be the linear functional defined by

$$\Lambda_{\hat{f}}(\hat{u}) = \int_{-\infty}^{\infty} \hat{u}(\xi + (-\tfrac{1}{2} + s)i)\overline{\hat{f}(\xi + (-\tfrac{1}{2} - s)i)}d\xi. \tag{12}$$

If $\hat{u} \in \mathcal{H}^{s}$

$$\begin{aligned}
|\Lambda_{\hat{f}}(\hat{u})| &\le \int_{-\infty}^{\infty} |\hat{u}(\xi + (-\tfrac{1}{2} + s)i)||\hat{f}(\xi + (-\tfrac{1}{2} - s)i)|d\xi \\
&\le \left(\int_{-\infty}^{\infty} (1+\xi^2)^{s}|\hat{u}(\xi + (-\tfrac{1}{2} + s)i)|^2 d\xi\right)^{1/2} \left(\int_{-\infty}^{\infty} (1+\xi^2)^{-s}|\hat{f}(\xi + (-\tfrac{1}{2} - s)i)|^2 d\xi\right)^{1/2} \\
&= |||\hat{u}|||_{s}|||\hat{f}|||_{-s}
\end{aligned}$$

so $\Lambda_{\hat{f}}(\hat{u})$ is well defined for $\hat{u} \in \mathcal{H}^{s}$ and $\Lambda_{\hat{f}}$ is a bounded linear functional on $\mathcal{H}$ with norm $\|\Lambda_{\hat{f}}\|_{(\mathcal{H}^{s})'} \le |||\hat{f}|||_{-s}$. In fact, we have $\|\Lambda_{\hat{f}}\|_{(\mathcal{H}^{s})'} = |||\hat{f}|||_{-s}$. To show this, let $\delta \in (0,1)$ and let

$$\hat{u}_{\delta}(\zeta) = [1 + (\zeta + (\tfrac{1}{2} - s - \delta)i)^2]^{-s}\hat{f}(\zeta - 2si).$$

Writing $\zeta = \xi + \eta i$, if $\eta \le -\frac{1}{2} + s$ the imaginary part of $\zeta + (\frac{1}{2} - s - \delta)i$ is $-(-\frac{1}{2} + s - \eta - \delta) \le 0$ so $\zeta \ne 0$ so the factor in front of $\hat{f}$ is well-defined and analytic in the half-space $\eta < -\frac{1}{2} + s$. Also the imaginary part of $\zeta - 2si$ is $\eta - 2s < -\frac{1}{2} - s$, so $\hat{f}(\zeta - 2si)$ is analytic in this half-space. Hence $\hat{u}_{\delta}$ is analytic in this half-plane and

$$\begin{aligned}
|||\hat{u}_{\delta}|||_{s}^{2} &= \int_{-\infty}^{\infty} (1+\xi^2)^{s}|\hat{u}_{\delta}(\xi + (-\tfrac{1}{2} + s)i)|^2 d\xi \\
&= \int_{-\infty}^{\infty} (1+\xi^2)^{s}|1 + (\xi - \delta i)^2|^{-2s}|\hat{f}(\xi + (-\tfrac{1}{2} - s)i)|^2 d\xi \\
&= \int_{-\infty}^{\infty} (1+\xi^2)^{s}[(1 + \xi^2 - \delta^2)^2 + 4\delta^2\xi^2]^{-s}|\hat{f}(\xi + (-\tfrac{1}{2} - s)i)|^2 d\xi \\
&\le C\int_{-\infty}^{\infty} (1+\xi^2)^{-s}|\hat{f}(\xi + (-\tfrac{1}{2} - s)i)|^2 d\xi = C|||\hat{f}|||_{-s}^{2}.
\end{aligned}$$

Also $\Lambda_{\hat f}(\hat u_\delta) = \int_{-\infty}^{\infty}[1+(\xi-\delta i)^2]^{-s}|\hat f(\xi+(-\frac{1}{2}-s)i)|^2 d\xi$. Hence

$$\begin{aligned}
\|\Lambda_{\hat f}\|_{(\mathcal{H}^s)'} &= \sup_{\hat u\in\mathcal{H}^2}\frac{\Lambda_{\hat f}(\hat u)}{|||\hat u|||_s} \\
&\geq \sup_{\delta>0}\frac{\int_{-\infty}^{\infty}[(1+\xi^2)^s[(1+\xi^2-\delta^2)^2+4\delta^2\xi^2]^{-s}|\hat f(\xi+(-\frac{1}{2}-s)i)|^2 d\xi}{\big[\int_{-\infty}^{\infty}[1+(\xi-\delta i)^2]^{-s}|\hat f(\xi+(-\frac{1}{2}-s)i)|^2 d\xi\big]^{1/2}} \\
&\geq \lim_{\delta\to 0}\frac{\int_{-\infty}^{\infty}[(1+\xi^2)^s[(1+\xi^2-\delta^2)^2+4\delta^2\xi^2]^{-s}|\hat f(\xi+(-\frac{1}{2}-s)i)|^2 d\xi}{\big[\int_{-\infty}^{\infty}[1+(\xi-\delta i)^2]^{-s}|\hat f(\xi+(-\frac{1}{2}-s)i)|^2 d\xi\big]^{1/2}} \\
&= \big[\int_{-\infty}^{\infty}[1+(\xi)^2]^{-s}|\hat f(\xi+(-\tfrac{1}{2}-s)i)|^2 d\xi\big]^{1/2} \\
&= |||\hat f|||_{-s}.
\end{aligned}$$

Hence $\|\Lambda_{\hat f}\|_{(\mathcal{H}^s)'} = |||\hat f|||_{-s}$. Finally we remark that if $\hat f \in \mathcal{H}^0$ then

$$\Lambda_{\hat f}(\hat u) = \int_{-\infty}^{\infty}\hat u(\xi-\tfrac{1}{2}i)\overline{\hat f(\xi-\tfrac{1}{2}i)}d\xi = (\hat u,\hat f)_{\mathcal{H}^0}. \tag{13}$$

The above arguments prove

Theorem 4. *If $s\in[0,1]$, the map $\hat f \to \Lambda_{\hat f}$ is an isomorphism from $(H^s(0,1))'$ to $\mathcal{H}^{-s}$. In particular, this map provides an identification of $H^{-1}(0,1)$ with $\mathcal{H}^{-1}$.*

Now we consider higher order spaces. Let $k>1$ be an integer and let $H_0^k(I)$ denote the set of functions in $H^k(I)$ with

$$D^j u(0) = D^j u(1) = 0 \text{ for } 0\leq j\leq k-1. \tag{14}$$

For $s=k+\sigma$ with $0<\sigma<1$, define $H^s(I)$ by interpolation,

$$H_0^s(I) = [H_0^{k+1}(I), H_0^k(I)]_{1-\sigma}.$$

If $0<\sigma<\frac{1}{2}$, then $H_0^s(I)$ is the closure in $H^s(I)$ of smooth functions which satisfy (14), while for $\frac{1}{2}<\sigma<1$, $H_0^s(I)$ is the closure in $H^s(I)$ of smooth functions which satisfy (14) plus the additional requirement $D^k u(0) = D^k u(1) = 0$. (The case $\sigma=\frac{1}{2}$ is special.) It is also true that if $u\in H_0^s(I)$, then $D^j u \in H^{s-j}(I)$ for $0\leq j\leq k$.

If $u\in H_0^s(I)$, we also denote by u the extension by 0 to $\mathbb{R}_+$. We have

Theorem 5. *$\mathcal{M}H_0^s(I) = \mathcal{H}^s$. If $u\in H_0^s(I)$, then*

$$\int_0^1 x^{-2s}u(x)^2 dx \leq C||u||_{H_0^s(I)} < \infty. \tag{15}$$

Proof. If $u\in H_0^s(I)$, then $u\in H_0^\sigma(I)$ so from Theorem 2, $\hat u(\zeta)$ is analytic for $\eta<\sigma-\frac{1}{2}$. Also, $D^k u\in H_0^\sigma(I)$, so $(\mathcal{M}D^k u)(\zeta) = (-1)^k(\zeta+i)\cdots(\zeta+ki)\hat u(\zeta+ki)\in\mathcal{H}^\sigma$. It is easily seen from this that $\hat u(\zeta)$ is analytic in the half space $\eta<s-\frac{1}{2}$, and that $|||\hat u|||_s<\infty$. Hence $\mathcal{M}H_0^s(I)\subset\mathcal{H}^s$. The converse inclusion is easily shown to hold, so we have equality of the two spaces. Finally, if $u\in H_0^s(I)$, then

$$\int_{-\infty}^{\infty}|\hat u(\xi+(s-\tfrac{1}{2})i|^2 d\xi \leq C||u||_{H_0^s(I)} < \infty.$$

Transforming this into the original variables, we get (15). ■

Let $s > \frac{1}{2}$, $s \neq$ integer $+\frac{1}{2}$, let $u \in H^s(\mathbb{R}_+)$, and suppose further that $u(x) \equiv 0$ for $x \geq 1$. In particular, $u \in H^\sigma(I)$ for any $\sigma < \frac{1}{2}$, so $\hat{u}(\zeta)$ is analytic for $\eta < 0$. It is of interest to consider the meromorphic extension of $\hat{u}$ to positive values of η, and to determine the poles of this extension. Let n be the *nearest* integer to s, and write $s = n + \sigma$ with $-\frac{1}{2} < \sigma < \frac{1}{2}$. Since $s > \frac{1}{2}$, $n > 0$. From the Sobolev imbedding theorem u is $(n-1)$ – times continuously differentiable on $[0, \infty)$, so the Taylor polynomial of u of degree $n-1$,

$$T_{n-1}(x) = \sum_{j=0}^{n-1} \frac{1}{j!} u^{(j)}(0) x^j,$$

is well defined. Let $R_{n-1} = u - T_{n-1}$ be the remainder, and let χ_I be the characteristic function of I. Thus, $u = \chi_I u = \chi_I T_{n-1} + \chi_I R_{n-1}$. As in §I.1.1, with some abuse of notation, we set

$$\hat{T}_{n-1}(\zeta) = \mathcal{M}\{\chi_I T_{n-1}\}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty T^*_{n-1}(\tau) e^{-i\zeta\tau} d\tau,$$

$$\hat{R}_{n-1}(\zeta) = \mathcal{M}\{\chi_I R_{n-1}\}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty R^*_{n-1}(\tau) e^{-i\zeta\tau} d\tau.$$

Using the formula

$$\mathcal{M}\{\chi_I x^k\}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-k\tau - i\zeta\tau} d\tau = \frac{-i}{\sqrt{2\pi}} \frac{1}{\zeta - ki}, \text{ for } \eta < k,$$

where the integral is absolutely convergent, we obtain

$$\hat{T}_{n-1}(\zeta) = \frac{-i}{\sqrt{2\pi}} \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} \frac{1}{\zeta - ki}, \text{ for } \eta < 0.$$

This formula provides a meromorphic extension of $\hat{T}_{n-1}(\zeta)$ to the complex plane. We then have

Theorem 6. *Let $s > \frac{1}{2}$, $s \neq$ integer $+\frac{1}{2}$. Let $u \in H^s(\mathbb{R}_+)$, and suppose further that $u(x) \equiv 0$ for $x \geq 1$. Then $\hat{R}_{n-1}(\zeta)$ is analytic in the half plane $\eta < s - \frac{1}{2}$, so the formula $\hat{u}(\zeta) = \hat{T}_{n-1}(\zeta) + \hat{R}_{n-1}(\zeta)$ provides a meromorphic extension of $\hat{u}$ to the half plane $\eta < s - \frac{1}{2}$. Furthermore,*

$$\int_0^1 x^{-2s} R_{n-1}(x)^2 dx \leq C||u||_s^2.$$

Proof. Let χ be a smooth function on $[0, \infty)$ which is $\equiv 1$ near 0 and $\equiv 0$ outside $(0, \frac{1}{2})$. Let $v = \chi R_{n-1}$. Then $v \in H^s(I)$, $v^{(k)}(0) = 0$ for $0 \leq k \leq n-1$, $v^{(k)}(1) = 0$ for $0 \leq k \leq n-1$. Since $s \neq$ integer $+\frac{1}{2}$, $v \in H_0^s(I)$ and from Theorem 5, $\hat{v} \in \mathcal{H}^s$, so $\hat{v}$ is analytic in the half plane $\eta < s - \frac{1}{2}$ and

$$\int_0^1 x^{-2s} v^2 dx \leq C||v||_s^2 \leq C||u||_s^2.$$

Hence

$$\int_0^\infty x^{-2s} (\chi_I R_{n-1})^2 dx \leq C||u||_s^2,$$

so

$$\int_0^\infty e^{2s\tau - \tau} R^*_{n-1}(\tau)^2 d\tau \leq C||u||_s^2.$$

Hence $e^{(s-1/2)\tau}\chi_{[0,\infty)}R^*_{n-1} \in L_2(\mathbb{R})$, so $\hat{R}_{n-1}(\zeta)$ is analytic in the half plane $\eta < s - \frac{1}{2}$, as asserted. ■

The extension formula of $\hat{u}(\zeta)$ given in Theorem 6 has a disadvantage: it does not provide a convenient way to take the inverse Mellin transform of $\hat{u}(\zeta)$ on lines $\Im\zeta = \eta > 0$ that do not contain a pole. We therefore give another formula for the meromorphic extension of $\hat{u}(\zeta)$ to $\eta < s - \frac{1}{2}$. Suppose, as above, that $u \in H^s(\mathbb{R}_+)$, and suppose further that for some $a < 1$, $u(x) \equiv 0$ for $x > a$. Let $\chi \in C_0^\infty(\mathbb{R})$ satisfy $\chi(x) \equiv 1$ for $0 < x < a$ and $\chi(x) \equiv 0$ for $x > 1$. Thus, $u = \chi u = \chi T_{n-1} + \chi R_{n-1}$. Setting

$$\hat{T}^{(1)}_{n-1}(\zeta) = \mathcal{M}\{\chi T_{n-1}\}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \chi^*(\tau)T^*_{n-1}(\tau)e^{-i\zeta\tau}d\tau,$$

$$\hat{R}^{(1)}_{n-1}(\zeta) = \mathcal{M}\{\chi R_{n-1}\}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \chi^*(\tau)R^*_{n-1}(\tau)e^{-i\zeta\tau}d\tau.$$

we have

$$\hat{u}(\zeta) = \hat{T}^{(1)}_{n-1}(\zeta) + \hat{R}^{(1)}_{n-1}(\zeta). \tag{16}$$

In order to use (16) to define a meromorphic extension of $\hat{u}(\zeta)$, a meromorphic extension of each of the terms on the right side of (16) must be given. One sees that $\chi R_{n-1} \in H_0^s(I)$, so $\mathcal{M}\{\chi R_{n-1}\}(\zeta)$ is analytic in the half-plane $\eta < s - \frac{1}{2}$. To understand the domain of analyticity of $\mathcal{M}\{\chi T_{n-1}\}(\zeta)$, it is convenient to make some definitions. For $x \geq 0$ define

$$\begin{aligned} a_k(x) &= \chi_I(x)x^k, \quad a_k^0(x) = (1-\chi_I(x))x^k,\\ b_k(x) &= \chi(x)x^k, \quad b_k^0(x) = (1-\chi(x))x^k,\\ c_k(x) &= (\chi_I(x) - \chi(x))x^k. \end{aligned}$$

One has

$$\hat{b}_k(\zeta) = \mathcal{M}\{b_k\}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \chi^*(\tau)e^{-(k-\eta)\tau}e^{-i\xi\tau}d\tau,$$

and the integral is absolutely convergent for $\eta < k$. Since $\chi_I(x) = \chi(x)$ outside of $(a, 1)$ we have, with $a^* = \ln a^{-1}$,

$$\hat{c}_k(\zeta) = \mathcal{M}\{(\chi_I - \chi)x^k\}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_0^{a^*} (1-\chi^*(\tau))e^{-(k-\eta)\tau}e^{-i\xi\tau}d\tau.$$

The integral defining $\hat{c}_k$ is absolutely convergent for all ζ, so $\hat{c}_k$ is an entire function. Also, an integration by parts shows that $\hat{c}_k(\zeta)$ satisfies the inequality

$$|\hat{c}_k(\zeta)| \leq C(\eta)|\xi|^{-1} \text{ as } |\xi| \to \infty.$$

A simple integration gives

$$\hat{a}_k(\zeta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(k+i\zeta)\tau}d\tau = \frac{-i}{\sqrt{2\pi}}\frac{1}{\zeta - ki}, \text{ for } \eta < k.$$

Since $b_k = a_k - c_k$,

$$\hat{b}_k(\zeta) = \frac{-i}{\sqrt{2\pi}}\frac{1}{\zeta - ki} - \hat{c}_k(\zeta), \text{ for } \eta < k. \tag{17}$$

Eqn (17) expresses $\hat{b}_k(\zeta)$ as a difference of a meromorphic function and an entire function in the half-plane $\eta < k$. Hence (17) provides a meromorphic continuation of $\hat{b}_k(\zeta)$ into the entire complex plane. The extended function, which we also denote by $\hat{b}_k(\zeta)$, has a single simple pole at $\zeta = ki$.

To gain more understanding of this meromorphic extension, we ask: is there a function whose Mellin transform equals $\hat{b}_k(\zeta)$ for $\eta > k$, and with the transform given by an absolutely convergent integral? The answer is yes, and the function is given by $b_k^0(x)$. To show this, note first that since $1 - \chi(x) = 0$ for $x < a$,

$$\hat{b}_k^0(\zeta) = \mathcal{M}\{b_k^0\}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a^*} (1 - \chi^*(\tau)) e^{(\eta - k)\tau} e^{-i\xi\tau} d\tau, \text{ for } \eta > k.$$

The integral defining $\hat{b}_k^0$ is absolutely convergent. Furthermore, since $\chi^*(a^*) = 1$ and $D_\tau^j \chi^*(a^*) = 0$ for $j = 1, 2, \cdots$, successive integrations by parts shows that $\hat{b}_k^0(\zeta)$ satisfies the inequality

$$|\hat{b}_k^0(\zeta)| \leq C(m, \eta)|\xi|^{-m} \text{ for } \eta > k, \text{ as } |\xi| \to \infty.$$

Similarly

$$\hat{a}_k^0(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-(k + i\zeta)\tau} d\tau = \frac{i}{\sqrt{2\pi}} \frac{1}{\zeta - ki}, \text{ for } \eta > k,$$

where the integral is absolutely convergent. Since $b_k^0 = -a_k^0 - c_k$,

$$\hat{b}_k^0(\zeta) = \frac{-i}{\sqrt{2\pi}} \frac{1}{\zeta - k} - \hat{c}_k(\zeta), \text{ for } \eta > k,$$

and this coincides with the meromorphic extension of $\hat{b}_k(\zeta)$ given in (17). Summarizing the above discussion, we have shown that

$$\hat{b}_k(\zeta) = \begin{cases} \mathcal{M}\{\chi(x)x^k\}(\zeta), & \text{for } \eta < k, \\ \mathcal{M}\{(1 - \chi(x))x^k\}(\zeta), & \text{for } \eta > k. \end{cases} \tag{18}$$

Using linearity, these formulas may be applied to the Taylor polynomial $T_{n-1}(x)$. The integral defining $\hat{T}_{n-1}^{(1)}(\zeta)$ is absolutely convergent for $\eta < 0$. Using (17), one obtains

$$\hat{T}_{n-1}^{(1)}(\zeta) = \frac{-i}{\sqrt{2\pi}} \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} \frac{1}{\zeta - ki} + \hat{S}_{n-1}(\zeta), \ \eta < 0, \tag{19}$$

where $\hat{S}_{n-1}(\zeta)$ is an entire function. Eqn (19) provides a meromorphic extension of $\hat{T}_{n-1}^{(1)}(\zeta)$ to the complex plane. The extended function, which we also denote by $\hat{T}_{n-1}^{(1)}(\zeta)$, has a simple pole at $\zeta = ki$, provided $u^{(k)}(0) \neq 0$. We obtain from (18) the formula

$$\hat{T}_{n-1}^{(1)}(\zeta) = \begin{cases} \mathcal{M}\{\chi(x)T_{n-1}(x)\}(\zeta), & \text{for } \eta < 0, \\ \mathcal{M}\{(1 - \chi(x))T_{n-1}(x)\}(\zeta), & \text{for } \eta > n - 1. \end{cases} \tag{20}$$

Also,

$$\hat{T}_{n-1}^{(1)}(\zeta) = \sum_{j=0}^{n-1} \frac{1}{j!} u^{(j)}(0) \hat{b}_j^0(\zeta), \text{ for } \eta > n - 1.$$

Since $s \neq$ integer $+\frac{1}{2}$, $|u^{(j)}(0)| \leq C||u||_s$, and $\hat{T}_{n-1}^{(1)}$ satisfies the inequality

$$|\hat{T}_{n-1}^{(1)}(\zeta)| \leq C(m, \eta)||u||_s|\xi|^{-m} \text{ for } \eta > n - 1, \text{ as } |\xi| \to \infty. \tag{21}$$

These formulas are used to prove the following theorem.

Theorem 7. *Let the hypotheses of Theorem 6 be satisfied. Let* $\hat{u}(\zeta)$ *be the meromorphic extension of* $\mathcal{M}\{u\}(\zeta)$ *to the half-plane* $\eta < s - \frac{1}{2}$*, so* $\hat{u}(\zeta)$ *is analytic everywhere in this half-plane except perhaps at the points* $\zeta = ki$*,* $k = 0, \cdots, n-1$*. Then* $\hat{u}(\cdot + (s - \frac{1}{2})i)$ *is square-integrable on* $\mathbb{R}$ *and, setting* $u_1^*(\tau) = \big(\mathcal{F}^{-1}\hat{u}(\cdot + (s - \frac{1}{2})i)\big)(\tau)$*,* $u_1^* \in H^s(\mathbb{R})$*. Also,*

$$||u_1^*||_{s,\mathbb{R}}^2 \le C(s) \int_{-\infty}^{\infty} (1+\xi^2)^s |\hat{u}(\xi + (s+\tfrac{1}{2})i)|^2 d\xi \le C_1(s)||u||_s^2. \tag{22}$$

Proof. We use the formula (16). From (21) we see that $\hat{T}_{n-1}^{(1)}$ is square-integrable on the line $\eta = s - \frac{1}{2}$, and $\mathcal{F}^{-1}\hat{T}_{n-1}^{(1)}(\cdot + (s+\frac{1}{2})i) \in H^s(\mathbb{R})$. As in the proof of Theorem 6, we see that $\hat{R}_{n-1}^{(1)} \in \mathcal{H}^s$, and $|||\hat{R}_{n-1}^{(1)}|||_s \le C||u||_s$. Hence, from the definition of the norm of $\mathcal{H}^s$, $\mathcal{F}^{-1}\hat{R}_{n-1}^{(1)}(\cdot + (s+\frac{1}{2})i) \in H^s(\mathbb{R})$ and

$$||\mathcal{F}^{-1}\hat{R}_{n-1}^{(1)}(\cdot + (s+\tfrac{1}{2})i)||_s = |||\hat{R}_{n-1}^{(1)}|||_s \le C||u||_s.$$

The inequality (22) then follows from (16) and the triangle inequality. ■