1.1 Meromorphic extension of a Mellin or Kondrat'yev transform

We consider a smooth function in a half line or a sector, and we determine the poles of the Mellin or Kondrat'yev transform, as well as the meromorphic extension of the transform to the complex plane.

Extension of the Mellin transform

Let $g \in C^{n+\sigma}([0,\infty))$; that is, g has continuous derivatives of order $\le n$ and $g^{(n)}$ is Holder continuous with exponent $\sigma \in (0,1]$. Suppose also that $g(x) \equiv 0$ for $x > 1$, so $g^*(\tau) \equiv 0$ for $\tau < 0$. We therefore may write

$$\hat{g}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g^*(\tau) e^{-i\zeta\tau} d\tau.$$

Since $\sup|g^*(\tau)| = \max|g(x)| < \infty$, the integral is absolutely convergent for $\eta < 0$. We are concerned with the meromorphic extension of $\hat{g}$ to values of $\eta > 0$.

Let T_n denote the Taylor polynomial of g of degree n, with remainder $R_n = g - T_n$, so

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(0) x^k.$$

Then $\max|R_n(x)| \le Cx^{n+\sigma}$, so

$$\sup|R_n^*(\tau)| \le Ce^{-(n+\sigma)\tau}. \tag{1}$$

Define

$$\hat{T}_n(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty T_n^*(\tau) e^{-i\zeta\tau} d\tau,$$

$$\hat{R}_n(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty R_n^*(\tau) e^{-i\zeta\tau} d\tau,$$

so

$$\hat{g}(\zeta) = \hat{T}_n(\zeta) + \hat{R}_n(\zeta). \tag{2}$$

Note that there is an abuse of notation here; $\hat{T}_n$ and $\hat{R}_n$ are not the Mellin transforms of T_n and R_n, since the above integrals are taken over $(0,\infty)$, not $(-\infty,\infty)$. In fact, $\hat{T}_n = \mathcal{M}\{\chi_I T_n\}$, $\hat{R}_n = \mathcal{M}\{\chi_I R_n\}$, where χ_I is the characteristic function of $I = [0,1]$.

From (1), the integral defining $\hat{R}$ is absolutely convergent for $\eta < n+\sigma$, and $\hat{R}_n$ is analytic in the half plane $\eta < n+\sigma$. Using the formula

$$\mathcal{M}\{\chi_I x^k\}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-k\tau - i\zeta\tau} d\tau = \frac{-i}{\sqrt{2\pi}} \frac{1}{\zeta - ki},$$

where the integral is absolutely convergent for $\eta < k$, we obtain

$$\hat{T}_n(\zeta) = \frac{-i}{\sqrt{2\pi}} \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} \frac{1}{\zeta - ki}.$$

We now return to the formula (2). The validity of this formula has been shown for $\eta < 0$, and all the functions are analytic in this half plane. The functions on the right are meromorphic in the region $\eta < n+\sigma$. We conclude that $\hat{g}(\zeta)$ has a meromorphic extension to the half plane $\eta < n+\sigma$, with simple poles at the points $\zeta = ki$, $k = 0, 1, \cdots, n$. If $g \in C_0^\infty([0,\infty))$, $\hat{g}$ is meromorphic in the entire complex plane with simple poles at $\zeta = ik$, k a non-negative integer.

Extension of the Kondrat'yev transform

Let $S \subset R^2$ be the sector of angle ω defined, in terms of polar coordinates, by $0 < \theta < \omega$. Let $f \in C^{n+\sigma}(\bar{S})$ with $f(x,y) \equiv 0$ for $r > 1$. Thus, $f^*(\tau,\theta) = 0$ for $\tau < 0$, and we may write

$$\hat{f}(\zeta,\theta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f^*(\tau,\theta) e^{-i\zeta\tau} d\tau.$$

Since $\sup |f^*(\tau,\theta)| = \max |f(x,y)| < \infty$, the integral is absolutely convergent for $\eta < 0$. We are concerned with the meromorphic extension of $\hat{f}$ to values of $\eta > 0$.

Let T_n denote the Taylor polynomial of f of degree n, with remainder $R_n = f - T_n$, so

$$T_n(x) = \sum_{0 \le k+m \le n} \frac{1}{k!m!} [D_{x_1}^k D_{x_2}^m f(0,0)] x_1^k x_2^m.$$

Then $\max |R_n(x,y)| \le C r^{n+\sigma}$, so $\sup |R_n^*(\tau,\theta)| \le C e^{-(n+\sigma)\tau}$. Define

$$\hat{T}_n(\zeta,\theta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty T_n^*(\tau,\theta) e^{-i\zeta\tau} d\tau,$$

$$\hat{R}_n(\zeta,\theta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty R_n^*(\tau,\theta) e^{-i\zeta\tau} d\tau,$$

so

$$\hat{f}(\zeta,\theta) = \hat{T}_n(\zeta,\theta) + \hat{R}_n(\zeta,\theta). \tag{3}$$

Again, there is an abuse of notation here: $\hat{T}_n$ and $\hat{R}_n$ are not the Kondrat'yev transforms of T_n and R_n. In fact, $\hat{T}_n = \mathcal{K}\{\chi_D T_n\}$, $\hat{R}_n = \mathcal{K}\{\chi_D R_n\}$, where χ_D is the characteristic function of the unit disk D. The integral defining $\hat{R}$ is absolutely convergent for $\eta < n + \sigma$, and $\hat{R}_n$ is analytic in the half plane $\eta < n + \sigma$. Using the formula

$$\mathcal{K}\{\chi_D x_1^k x_2^m\}(\zeta,\theta) = \frac{\cos^k\theta \sin^m\theta}{\sqrt{2\pi}} \int_0^\infty e^{-(k+m)\tau - i\zeta\tau} d\tau = \frac{-i\cos^k\theta \sin^m\theta}{\sqrt{2\pi}} \frac{1}{\zeta - (k+m)i},$$

where the integral is absolutely convergent for $\eta < k + m$, we obtain

$$\hat{T}_n(\zeta,\theta) = \frac{-i}{\sqrt{2\pi}} \sum_{0 \le k+m \le n} \frac{\cos^k\theta \sin^m\theta}{k!m!} [D_{x_1}^k D_{x_2}^m f(0,0)] \frac{1}{\zeta - (k+m)i}.$$

The functions on the right side of (3) are meromorphic in the half plane $\eta < n + \sigma$, and $\hat{f}$ is analytic in the half plane $\eta < 0$. We conclude that $\hat{f}(\zeta,\theta)$ has a meromorphic extension to the half plane $\eta < n + \sigma$, with simple poles at the points $\zeta = ki$, $k = 0, 1, \cdots, n$. If $f \in C_0^\infty(R^2)$, $\hat{f}(\zeta,\theta)$ is meromorphic in the entire complex plane with simple poles at $\zeta = ik$, k a non-negative integer.