

1. The tools

We start with some facts and formulas that will be used in the analysis.

Fourier transform

A good reference is [Yosida, Chapter 6]. Let u be defined on $\mathbb{R} = (-\infty, \infty)$. We denote the Fourier transform of u by $\mathcal{F}u$ or \hat{u} . The basic formulas are

$$\begin{aligned}\hat{u}(\xi) &= (\mathcal{F}u)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x) dx, \\ u(x) &= (\mathcal{F}^{-1}\hat{u})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \hat{u}(\xi) d\xi, \\ \int_{-\infty}^{\infty} u(x) \overline{v(x)} dx &= \int_{-\infty}^{\infty} \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi, \\ \mathcal{F}(u')(x) &= i\xi \hat{u}(\xi), \\ \mathcal{F}(u(\cdot + \beta))(x) &= e^{i\beta x} \hat{u}(x), \\ \mathcal{F}(u * v)(x) &= \sqrt{2\pi} \hat{u}(x) \hat{v}(x),\end{aligned}$$

where $(u * v)(x) = \int_{-\infty}^{\infty} u(s)v(x - s)ds$. For transforms in the complex plane, with $\zeta = \xi + i\eta$ we have

$$\begin{aligned}\hat{u}(\zeta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta x} u(x) dx, \\ u(x) &= \frac{1}{\sqrt{2\pi}} \int_{\xi=-\infty}^{\infty} e^{i\zeta x} \hat{u}(\zeta) d\xi, \\ \mathcal{F}(e^{\beta x} u(x))(\zeta) &= \hat{u}(\zeta + \beta i), \\ \mathcal{F}(e^{\beta x} u'(x))(\zeta) &= i(\zeta + \beta i) \hat{u}(\zeta + \beta i), \\ \hat{u}'(\zeta) &= -i\mathcal{F}(xu(x))(\zeta).\end{aligned}$$

Mellin transform

The Mellin transform of a function is defined by first making a change of variable and then applying the Fourier transform. Let u be defined on $\mathbb{R}_+ = (0, \infty)$. Consider the change of variable

$$\begin{aligned}x &= e^{-\tau} \\ \tau &= -\ln x\end{aligned}$$

so

$$dx = -e^{-\tau} d\tau, \quad u_x = -e^{\tau} u_{\tau}^*, \quad u_{xx} = e^{2\tau} [u_{\tau\tau}^* + u_{\tau}^*].$$

x	τ
0	∞
1	0
∞	$-\infty$

We use the notation u^* to express this change of variables, and we write $u(x) = u^*(\tau)$. Let $\hat{u}(\zeta) = \mathcal{F}(u^*(\tau))(\zeta)$ be the Fourier transform of u^* with respect to τ . Then $\mathcal{M}u = \hat{u}$ is the Mellin transform of u . Thus, the Mellin transform is given by the formula

$$\begin{aligned}\mathcal{M}(u)(\zeta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta\tau} u^*(\tau) d\tau = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{i\zeta-1} u(x) dx, \\ \mathcal{M}^{-1}(\hat{u})(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\zeta\tau} \hat{u}(\zeta) d\zeta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{-i\zeta} \hat{u}(\zeta) d\zeta.\end{aligned}$$

We have

$$\begin{aligned}(\mathcal{M}u_x)(\zeta) &= -i(\zeta + i)\hat{u}(\zeta + i), \\ (\mathcal{M}u_{xx})(\zeta) &= -(\zeta + 2i)(\zeta + i)\hat{u}(\zeta + 2i).\end{aligned}$$

If $u \in L_2(0, \infty)$, we have

$$\int_0^{\infty} u(x)^2 dx = \int_{-\infty}^{\infty} u^*(\tau)^2 e^{-\tau} d\tau = \int_{-\infty}^{\infty} |\hat{u}(\xi - \frac{1}{2}i)|^2 d\xi < \infty.$$

If $u_x \in L_2(0, \infty)$, we have

$$\int_0^{\infty} u_x(x)^2 dx = \int_{-\infty}^{\infty} u_x^*(\tau)^2 e^{\tau} d\tau = \int_{-\infty}^{\infty} |\xi + \frac{1}{2}i|^2 |\hat{u}(\xi + \frac{1}{2}i)|^2 d\xi < \infty,$$

If $x^{-1}u \in L_2(0, \infty)$, we have

$$\int_0^{\infty} x^{-2}u(x)^2 dx = \int_{-\infty}^{\infty} u^*(\tau)^2 e^{\tau} d\tau = \int_{-\infty}^{\infty} |\hat{u}(\xi + \frac{1}{2}i)|^2 d\xi < \infty.$$

Hence, if u_x and $x^{-1}u \in L_2(0, \infty)$ and if $u \neq 0$, we have

$$\begin{aligned}\int_0^{\infty} x^{-2}u(x)^2 dx &= 4 \cdot \int_{-\infty}^{\infty} \frac{1}{4} |\hat{u}(\xi + \frac{1}{2}i)|^2 d\xi \\ &< 4 \int_{-\infty}^{\infty} |\xi + \frac{1}{2}i|^2 |\hat{u}(\xi + \frac{1}{2}i)|^2 d\xi \\ &= 4 \int_0^{\infty} u_x(x)^2 dx,\end{aligned}$$

which is Hardy's inequality. To be precise, Hardy's inequality contains the additional assertion that if $u' \in L_2(0, \infty)$ and if $u(0) = 0$, then $x^{-1}u \in L_2(0, \infty)$. A proof of this is given below.

Kondrat'yev transform

We use the term "Kondrat'yev transform" to denote the Mellin transform applied in a two dimensional setting. If $u(x)$ is a function of 2 variables, let (r, θ) denote the corresponding polar coordinates and we write $u^\#(r, \theta) = u(r \cos \theta, r \sin \theta)$. Make the change of independent variable $r = \exp(-\tau)$, $\tau = -\ln r$ as above, and write $u^*(\tau, \theta) = u^\#(r, \theta)$. Let $\hat{u}(\zeta, \theta) = \mathcal{F}(u^*(\tau, \theta))(\zeta, \theta)$ be the Fourier transform of u^* with respect to τ . We define $\mathcal{K}u = \hat{u}$ to be the *Kondrat'yev transform* of u . Thus, the Kondrat'yev transform is defined by the formula

$$\begin{aligned}\mathcal{K}(u)(\zeta, \theta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta\tau} u^*(\tau, \theta) d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} r^{i\zeta-1} u(r \cos \theta, r \sin \theta) dr.\end{aligned}$$

The following formulas are often useful in connection with the Kondrat'yev transform.

$$\begin{aligned}
|D^1 u|^2 &= u_x^2 + u_y^2 = u_r^2 + r^{-2} u_\theta^2 = e^{2\tau} [u_\tau^{*2} + u_\theta^{*2}], \\
|D^2 u|^2 + r^{-2} |D^1 u|^2 &\sim u_{rr}^2 + r^{-2} u_{r\theta}^2 + r^{-4} u_{\theta\theta}^2 + r^{-2} u_r^2 + r^{-4} u_\theta^2, \\
\Delta u &= u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta} = e^{2\tau} [u_{\tau\tau}^* + u_{\theta\theta}^*], \\
\int \int_S u^2 dx &= \int_{\tau=-\infty}^{\infty} \int_0^\omega u^*(\tau, \theta)^2 e^{-2\tau} d\tau d\theta \\
&= \int_{\xi=-\infty}^{\infty} \int_0^\omega |\hat{u}(\xi - i, \theta)|^2 d\xi d\theta, \\
\int \int_S r^{-2\beta} u^2 dx &= \int_{\tau=-\infty}^{\infty} \int_0^\omega u^*(\tau, \theta)^2 e^{2(\beta-1)\tau} d\tau d\theta \\
&= \int_{\xi=-\infty}^{\infty} \int_0^\omega |\hat{u}(\xi + (\beta-1)i, \theta)|^2 d\xi d\theta, \\
\int \int_S |D^1 u|^2 dx &= \int_0^\omega \int_{-\infty}^{\infty} [u_\tau^{*2} + u_\theta^{*2}] d\tau d\theta \\
&= \int_0^\omega \int_{-\infty}^{\infty} [\xi^2 |\hat{u}(\xi, \theta)|^2 + |\hat{u}_\theta(\xi, \theta)|^2] d\xi d\theta.
\end{aligned}$$

In the integrals, S denotes the sector with angle ω defined by the inequality $0 < \theta < \omega$.

We obtain from these formulas an inequality analogous to Hardy's inequality given above. Let $u \in H^1(S)$, and suppose further that $u = 0$ on a side of Γ ; say, $u(x_1, 0) = 0$. Then $\hat{u}(\zeta, 0) = 0$. From consideration of the eigenvalue problem $z''(\theta) + \lambda z(\theta) = 0$ with boundary conditions $z(0) = z'(\omega) = 0$ one obtains the inequality

$$\int_0^\omega |\hat{u}(\xi, \theta)|^2 d\theta \leq \frac{4\omega^2}{\pi^2} \int_0^\omega |\hat{u}_\theta(\xi, \theta)|^2 d\theta,$$

so one obtains the "Poincaré inequality" for the sector:

$$\begin{aligned}
\int \int_S r^{-2} u^2 dx &= \int_{-\infty}^{\infty} \int_0^\omega |\hat{u}(\xi, \theta)|^2 d\theta d\xi \\
&\leq \frac{4\omega^2}{\pi^2} \int_{-\infty}^{\infty} \int_0^\omega |\hat{u}_\theta(\xi, \theta)|^2 d\theta d\xi \\
&\leq \frac{4\omega^2}{\pi^2} \int \int_S |\nabla u|^2 dx.
\end{aligned}$$

If $u = 0$, on both sides of S , the inequality can be improved. We obtain

$$\int \int_S r^{-2} u^2 dx \leq \frac{\omega^2}{\pi^2} \int \int_S |\nabla u|^2 dx \text{ for } u \in H_0^1(S).$$

It will be seen, later, that such inequalities are true with considerably weaker assumptions on u ; u need only lie in a certain space, arising from the interpolation of two Sobolev spaces. On the other hand, the inequality does not hold for any smooth function in \bar{S} which vanishes in a neighborhood of the origin and which vanishes for $r > 1$. The reason is that this class of functions includes the functions $z(r)$ which vanish near $r = 0$ and for $r > 1$. The inequality in question then becomes $\int_0^1 r^{-1} z(r)^2 dr \leq C \int_0^1 r z_r(r)^2 dr$. Upon changing to τ , this inequality reads $\int_0^\infty z^*(\tau)^2 d\tau \leq C \int_0^\infty z_\tau^*(\tau)^2 d\tau$, for functions z^* which vanish near $\tau = 0$ and $\tau = \infty$. This latter inequality is false, so the above inequality does not hold for such functions u , with

a constant C independent of u . (In fact, this latter, invalid, inequality is exactly the inequality [HLP, 330] in the critical case, when it is not valid.) Our conclusion is that the above displayed inequality *requires* the vanishing of u on at least one of the sides of S .

Paley Wiener theorem

Let a and b be real numbers with $a < b$. If $e^{ax}u(x) \in L_2(\mathbb{R})$ and $e^{bx}u(x) \in L_2(\mathbb{R})$, then for $a < \eta < b$, $e^{\eta x}u(x) \in L_2(\mathbb{R})$ the integral defining $\hat{u}(\xi + \eta i)$ is absolutely convergent and $\hat{u}(\zeta)$ is an analytic function in the strip $a < \eta < b$. Furthermore,

$$(1) \quad \begin{aligned} \int_{-\infty}^{\infty} |u(x)|^2 e^{2ax} dx &= \int_{-\infty}^{\infty} |\hat{u}(\xi + ia)|^2 d\xi = \lim_{\eta \rightarrow a+0} \int_{-\infty}^{\infty} |\hat{u}(\xi + i\eta)|^2 d\xi, \\ \int_{-\infty}^{\infty} |u(x)|^2 e^{2bx} dx &= \int_{-\infty}^{\infty} |\hat{u}(\xi + ib)|^2 d\xi = \lim_{\eta \rightarrow b-0} \int_{-\infty}^{\infty} |\hat{u}(\xi + i\eta)|^2 d\xi. \end{aligned}$$

Also, from Schwarz's inequality we have the *three lines inequality*, valid for $0 \leq \theta \leq 1$,

$$(2) \quad \int_{-\infty}^{\infty} |\hat{u}(\xi + ((1-\theta)a + \theta b)i)|^2 d\xi \leq \left[\int_{-\infty}^{\infty} |\hat{u}(\xi + ai)|^2 d\xi \right]^{1-\theta} \left[\int_{-\infty}^{\infty} |\hat{u}(\xi + bi)|^2 d\xi \right]^{\theta}.$$

Conversely, if $\hat{u}(\zeta)$ is analytic in the strip $a < \eta < b$, and if $\int |\hat{u}(\xi + i\eta)|^2 d\xi < \infty$ for $a \leq \eta \leq b$, then for $a < \eta < b$ the inverse transform

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\Im \zeta = \eta} e^{i\zeta x} \hat{u}(\zeta) d\xi$$

is absolutely convergent and independent of η . Furthermore, (1) and (2) hold. If $a \leq \alpha, \beta \leq b$ and $\alpha + \beta = 2\eta$, then $\int e^{\alpha x} u(x) e^{\beta x} \overline{v(x)} dx = \int e^{\eta x} u(x) e^{\eta x} \overline{v(x)} dx$. Taking the Fourier transform of this equality, we obtain

$$(3) \quad \int_{-\infty}^{\infty} \hat{u}(\xi + \alpha i) \overline{\hat{v}(\xi + \beta i)} d\xi = \int_{-\infty}^{\infty} \hat{u}(\xi + \eta i) \overline{\hat{v}(\xi + \eta i)} d\xi, \quad \alpha + \beta = 2\eta.$$

If $u \in L_2(\mathbb{R})$ and $u(x) \equiv 0$ for $x < 0$, the integral defining $\hat{u}(\zeta)$ is absolutely convergent and \hat{u} is analytic in the half plane $\eta < 0$. Furthermore, for $\eta < 0$, $\int |\hat{u}(\xi + i\eta)|^2 d\xi$ is an increasing function of η and

$$\int_0^{\infty} |u(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 d\xi = \lim_{\eta \rightarrow -0} \int_{-\infty}^{\infty} |\hat{u}(\xi + i\eta)|^2 d\xi.$$

Conversely, if $\hat{u}(\zeta)$ is analytic for $\eta < 0$, and if

$$\sup_{\eta < 0} \int_{-\infty}^{\infty} |\hat{u}(\xi + i\eta)|^2 d\xi < \infty,$$

then $u(x) = (\mathcal{F}^{-1}\hat{u})(x)$ vanishes for $x < 0$. If $v \in L_2(\mathbb{R})$ and $v(x) \equiv 0$ for $x < x_1$, setting $u(x) = v(x + x_1)$, we have $u \in L_2(\mathbb{R})$, $u(x) \equiv 0$ for $x < 0$, and $\hat{v}(\zeta) = e^{-ix_1\zeta} \hat{u}(\zeta)$. Hence $\hat{v}(\zeta)$ is analytic for $\eta < 0$ and

$$\sup_{\eta < 0} e^{-2x_1\eta} \int_{-\infty}^{\infty} |\hat{v}(\xi + i\eta)|^2 d\xi < \infty.$$

If $u \in L_2(\mathbb{R})$ we may write $u = v + w$ where $v(x) \equiv 0$ for $x < 0$ and $w(x) \equiv 0$ for $x > 0$. The functions v and w are orthogonal and $v = Pu$ where P is the orthogonal projection of $L_2(\mathbb{R})$ onto $L_2(\mathbb{R}_+)$. It is of importance to determine the Fourier transform of this projection. We have $\hat{u} = \hat{v} + \hat{w}$ where $\hat{v}(\zeta)$ is analytic for $\eta < 0$ and $\hat{w}(\zeta)$ is analytic for $\eta > 0$. For $\varepsilon > 0$,

$$\hat{v}(\xi' - \varepsilon i) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(\varepsilon + \xi' i)x} v(x) dx.$$

Let

$$\chi(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-(\varepsilon + \xi' i)x}, & x > 0, \\ 0, & x < 0, \end{cases}$$

so $\chi \in L_2(\mathbb{R})$ and its the Fourier transform is given by

$$\hat{\chi}(\xi) = \frac{1}{2\pi(\varepsilon + (\xi - \xi')i)}.$$

By Parseval's formula

$$\begin{aligned} \hat{v}(\xi' - \varepsilon i) &= \int_{-\infty}^{\infty} v(x) \overline{\chi(x)} dx \\ &= \int_{-\infty}^{\infty} \hat{v}(\xi) \overline{\hat{\chi}(\xi)} d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{v}(\xi)}{\varepsilon + (\xi' - \xi)i} d\xi. \end{aligned}$$

In other words,

$$(4) \quad \hat{v}(\zeta') = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{v}(\xi)}{\zeta' - \xi} d\xi, \quad \text{if } v \in L_2(\mathbb{R}) \text{ and } v(x) \equiv 0 \text{ for } x < 0.$$

Since $\hat{w}(\zeta)$ is analytic for $\eta > 0$, the Cauchy integral theorem may be used to show that

$$\int_{-\infty}^{\infty} \frac{\hat{w}(\xi)}{\varepsilon + (\xi' - \xi)i} d\xi = 0, \quad \varepsilon > 0.$$

Hence we obtain

$$(5) \quad \hat{v}(\xi') = \frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{\hat{u}(\xi)}{\varepsilon + (\xi' - \xi)i} d\xi.$$

Equation (5) gives the desired formula for the Fourier transform of P .

Hardy's inequality

We use the Mellin transform to give a proof of Hardy's inequality.

Theorem 1. *Let $u \in H^1(\mathbb{R}_+)$ with $u(0) = 0$. Then $x^{-1}u \in L_2(\mathbb{R}_+)$ and if $u \neq 0$,*

$$(6) \quad \int_0^{\infty} x^{-2} u(x)^2 dx < 4 \int_0^{\infty} u'(x)^2 dx.$$

Proof. Since u may be approximated in $H^1(\mathbb{R}_+)$ by functions with compact support, we may suppose that $u(x) \equiv 0$ for $x > x_0$. Since the inequality remains unchanged under a change of scale, we may replace $u(x)$ by $u(x/x_0)$ and suppose that $u(x) \equiv 0$ for $x > 1$. Since $u \in L_2(\mathbb{R}_+)$, we find that $\hat{u}(\zeta)$ is analytic for $\eta < -\frac{1}{2}$ and satisfies

$$\sup_{\eta \leq -1/2} \int |\hat{u}(\xi + i\eta)|^2 d\xi < \infty.$$

Similarly, since $u' \in L_2(\mathbb{R}_+)$, we find that $\zeta \hat{u}(\zeta)$ is analytic for $\eta < \frac{1}{2}$ and satisfies

$$\sup_{\eta \leq 1/2} \int |\xi + i\eta|^2 |\hat{u}(\xi + i\eta)|^2 d\xi < \infty.$$

Hence \hat{u} is analytic for $\eta < \frac{1}{2}$ except possibly for a simple pole at $\zeta = 0$. Let $\hat{u} = a/\zeta + \hat{z}$ where \hat{z} is analytic for $\eta < \frac{1}{2}$. Then \hat{z} satisfies

$$\sup_{\eta \leq -1/2} \int |\hat{z}(\xi + i\eta)|^2 d\xi < \infty,$$

$$\int |\hat{z}(\xi + i\eta)|^2 d\xi < \infty \text{ for each } \eta \neq 0.$$

By the three lines inequality (2), we have for $-\frac{1}{2} < \eta < \frac{1}{2}$,

$$\int_{-\infty}^{\infty} |\hat{z}(\xi + \eta i)|^2 d\xi \leq \left[\int_{-\infty}^{\infty} |\hat{z}(\xi - \frac{1}{2}i)|^2 d\xi \right]^{1/2-\eta} \left[\int_{-\infty}^{\infty} |\hat{z}(\xi + \frac{1}{2}i)|^2 d\xi \right]^{1/2+\eta}.$$

Hence

$$\sup_{\eta \leq 1/2} \int |\hat{z}(\xi + i\eta)|^2 d\xi < \infty.$$

Hence $z = \mathcal{M}^{-1}\hat{z}$ is a function which vanishes for $x > 1$ and for which

$$\int_0^{\infty} x^{-2} z(x)^2 dx = \int_0^{\infty} |\hat{z}(\xi + \frac{1}{2}i)|^2 d\xi < \infty.$$

A computation shows that $\mathcal{M}\{1\}$ is analytic for $\eta < 0$ and satisfies $\mathcal{M}\{1\}(\zeta) = -i/(\sqrt{2\pi}\zeta)$. Hence

$$u(x) = \sqrt{2\pi}ai + z(x).$$

Since u also vanishes for $x > 1$, we have $a = 0$ and $u = z$, so $x^{-1}u \in L_2(\mathbb{R}_+)$. The proof of (6) is given above. ■

Weighted inequality in two dimensions

Let $S \subset \mathbb{R}^2$ be the sector of angle ω defined, in terms of polar coordinates, by $0 < \theta < \omega$. We use the Kondrat'yev transform to give a proof of the following weighted inequality. An improved form of the inequality will be given in Lemma 3;9.

Theorem 2. *Let $0 < \beta < 1$. Then there is a constant $C_\beta > 0$ such that if $u \in H^1(S)$, then*

$$\int_S r^{-2\beta} |u(x)|^2 dx \leq C_\beta \|u\|_{1,S}^2.$$

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^2)$ with $\chi \equiv 1$ for $r < \frac{1}{2}$, $\chi \equiv 0$ for $r > 1$. Let $u \in H^1(S)$, and let $v = \chi u$. Let S^* be the strip in the $\tau\theta$ plane defined by the inequalities $-\infty < \tau < \infty$, $0 < \theta < \omega$. Then $e^{-\tau}v^* \in L_2(S^*)$, v^* has weak derivatives v_τ^* and v_θ^* with $v_\tau^*, v_\theta^* \in L_2(S^*)$, and

$$\|v\|_{1,S}^2 = \int_{-\infty}^{\infty} \int_0^\omega [e^{-2\tau} v^*(\tau, \theta)^2 + v_\tau^*(\tau, \theta)^2 + v_\theta^*(\tau, \theta)^2] d\tau d\theta.$$

Since $v = 0$ for $r \geq 1$, $v^* \equiv 0$ for $\tau < 0$, and from the Paley-Wiener theorem, $\hat{v}(\zeta, \theta)$ is analytic for $\eta < -1$. Similarly, $\mathcal{F}(v_\tau^*)(\zeta, \theta)$ and $\mathcal{F}(v_\theta^*)(\zeta, \theta)$ are analytic for $\eta < 0$. Since v_τ^* is the weak derivative of v^* , $\mathcal{F}(v_\tau^*)(\zeta, \theta) = i\zeta \hat{v}(\zeta, \theta)$. Similarly, $\mathcal{F}(v_\theta^*)(\zeta, \theta) = \hat{v}_\theta(\zeta, \theta)$ is the weak derivative of $\hat{v}(\zeta, \theta)$. Hence $\zeta \hat{v}(\zeta, \theta)$ has an analytic extension from the half plane $\eta < -1$ to $\eta < 0$, so $\hat{v}(\zeta, \theta)$ has a meromorphic extension to $\eta < 0$ with at most a simple pole at $\zeta = 0$. Also, \hat{v} has a weak derivative \hat{v}_θ in $\eta < 0$, and

$$\sup \left\{ \int_{-\infty}^{\infty} \int_0^\omega [|\hat{v}(\xi + (\eta - 1)i, \theta)|^2 + |\xi + \eta i|^2 |\hat{v}(\xi + \eta i, \theta)|^2 + |\hat{v}_\theta(\xi + \eta i, \theta)|^2] d\xi d\theta \right\} \|v\|_{1,S}^2$$

$$\leq C \|u\|_{1,S}^2 < \infty.$$

In particular,

$$\int_{-\infty}^{\infty} \int_0^{\omega} |\hat{v}(\xi + (\beta - 1)i, \theta)|^2 d\xi d\theta \leq (1 - \beta)^{-2} \int_S |D^1 v|^2 dx.$$

Transforming this inequality,

$$(6) \quad \int_S r^{-2\beta} v^2 dx \leq (1 - \beta)^{-2} \int_S |D^1 v|^2 dx.$$

Since $v = u$ for $r < \frac{1}{2}$,

$$\int_0^{1/2} \int_0^{\omega} r^{-2\beta} u^2 dx \leq (1 - \beta)^{-2} \|v\|_{1,S}^2 \leq C(1 - \beta)^{-2} \|u\|_{1,S}^2.$$

Since

$$\int_{1/2}^{\infty} \int_0^{\omega} r^{-2\beta} u^2 dx \leq 2^{2\beta} \int_S u^2 dx \leq 2^{2\beta} \|u\|_{1,S}^2,$$

we obtain the result. ■

The inequality is not valid with $\beta = 1$ unless u satisfies some more conditions. For example, if $u \in H_0^1(S)$, or if u vanishes on a side of S , the inequality with $\beta = 1$ is proved above. The weakest condition on $u \in H^1(S)$ that guarantees $\int_S r^{-2} u^2 dx < \infty$ is that u belong to the interpolation space $[\tilde{H}_0^{1+\varepsilon}(S), H^{1-\varepsilon}(S)]_{1/2}$ for some $\varepsilon \in (0, 1)$, where $\tilde{H}^{1+\varepsilon}(S)$ is the closed subspace of $H^{1+\varepsilon}(S)$ consisting of functions which vanish at $x = 0$.