Corner singularities and singular perturbations

R. Bruce Kellogg*

Sommario – Facciamo una singolarità del angolo espansione per una singolarmente perturbato equazione ellittica nel settore. Risultato usa ottenere i limites di derivati di soluzione. I nostri limites di derivati mostrano ambedue il frontiere strato et la singolarità del angolo.

Abstract – A corner singularity expansion is developed for a singularly perturbed elliptic boundary value problem. The problem is set in a sector of the plane. In the expansion, particular attention is paid to the singular perturbation parameter. The result is used to give pointwise bounds on derivatives of the solution. These bounds show the influence of both the boundary layers and the corner singularity.

1. Introduction The theory of corner singularities for an elliptic boundary value problem has been developed in the books of Grisvard [3,4], Dauge [2], Kozlov, Maz'ya, and Rossmann [6], and Nazarov and Plamenevskii [7]. In these works attention is paid to the highest order terms in the elliptic operator, which determines the singularities in the solution that come from a corner of the boundary. However the lower order terms also have an effect on the singularity expansion of a solution. This is particularly true for a singularly perturbed problem, in which the highest order terms are multiplied by a small parameter. In this note we consider the singularly perturbed convection diffusion problem

$$Lu := -\varepsilon \Delta u + p u_x + q u = f \text{ in } S,\ u = 0 \text{ on } \Gamma = \partial S. \tag{1.1}$$

The region S is the sector in R^2 defined by

$$S = \{(r\cos\theta, r\sin\theta) : r > 0,\ \omega_1 < \omega < \omega_2\},$$

where ω_l, $l = 1, 2$, are two numbers satisfying $\omega_1 < \omega_2$. The angle of the sector is $\omega = \omega_2 - \omega_1$. We assume that $\omega \in (0, 2\pi]$. Some more restrictions on the angles will be made as needed. We also set $\alpha = \pi/\omega$. The two rays that make up the boundary Γ of S are denoted Γ_l, $l = 1, 2$. The coefficients ε, p and q are taken to be positive constants with $\varepsilon \in (0, 1]$.

It is known that solutions to problems of the form (1.1) have singularities at the vertex of S. In the case of the problem $\Delta u = f$ in S, $u = 0$ on Γ, the singularities are expressed by the "singular functions" $z_j = r^{j\alpha} \sin j\alpha(\theta - \omega_1)$. The same functions z_j could also be used as the basis of a singularity expansion for the problem (1.1). However we will show that it is more convenient to use singular functions that are defined in terms of the modified Bessel functions $I_{j\alpha}$ and $K_{j\alpha}$. (Recall that $I_{j\alpha}(r)$ behaves like $r^{j\alpha}$ at the origin.) An expansion of the solution of (1.1) into these singular functions plus a smoother remainder is given in Section 2. Particular attention is paid to formulas and bounds for the associated linear functionals, especially the dependence of these bounds on the parameter ε.

Section 3 contains an application of this expansion. We treat the case of convex "outgoing" sector, whose angles satisfy $\omega = \omega_2 - \omega_1 < \pi$ and $0 < \omega_1 < \pi < \omega_2 < 2\pi$. In this case, the convective vector $[p, 0]^T$ points out of S. (See the figure in Section 3.) Here one expects a boundary layer, and the principal facts regarding this boundary layer are established. The case of an "incoming" sector is rather lengthy and will be treated in another paper.

We shall use the sector S and the truncated sector S_a, truncated at radius $a > 0$. The norm of a function u in the Sobolev space $H^s(S)$ is denoted $\|u\|_s$; the norm of $u \in H^s(\Omega)$ where Ω is a set other than S is denoted $\|u\|_{s,\Omega}$.

---

* Indirizzo dell'autore: Box 698, Landrum, SC 29356, USA. E-mail: kellogg@ipst.umd.edu

2. The corner singularity expansion

In this section we develop a corner singularity expansion for the problem (1.1) that pays particular attention to the paramter ε. We first note the existence of a solution and some associated basic inequalities.

Lemma 1. *Suppose $f \in H^{-1}(S)$. The problem (1.1) has a weak solution $u \in H_0^1(S)$, satisfying*

$$\varepsilon\|u\|_1 + \varepsilon^{1/2}\|u\|_0 \le C\|f\|_{-1}.$$

If $f \in L_2(S)$, then

$$\varepsilon^{1/2}\|u\|_1 + \|u\|_0 \le C\|f\|_0. \tag{2.1}$$

Suppose $f \in H^{k-2}(S)$ for some integer $k \ge 2$. Let $S' \subset S$ be a subregion with smooth boundary. Then $u \in H^k(S')$ and

$$\|u\|_{k,S'} \le C\varepsilon^{-(k-1/2)}\|f\|_{k-2}.$$

In particular, the boundary of S' may include portions of the boundary of S. Finally, if $f \in H^{s-2}(S)$ for some real number $s > 1$, then

$$\begin{aligned} \|u\|_1 &\le C\varepsilon^{-\frac{1}{2}(3-s)}\|f\|_{s-2},\ 1 \le s \le 2,\\ \|u\|_{s,S'} &\le C\varepsilon^{-(s-\frac{1}{2})}\|f\|_{s-2},\ s \ge 2. \end{aligned} \tag{2.2}$$

Proof. From the definition of a weak solution one has $\varepsilon|u|_1^2 + q\|u\|_0^2 = \int_\Omega uf \le \|f\|_{-1}\|u\|_1$. From this one obtains $\varepsilon\|u\|_1 + \varepsilon^{1/2}\|u\|_0 \le C\|f\|_{-1}$, and in a similar way, if $f \in L_2(S)$, one obtains $\varepsilon^{1/2}\|u\|_1 + \|u\|_0 \le C\|f\|_0$. This gives the first two assertions of the lemma. The regularity assertion follows from standard elliptic theory. For the interior estimate, let χ be a smooth function which is 1 on S' and which vanishes on a subregion S'' which has smooth boundary and lies between S and S'. Let $u' = \chi u$. Then u' satisfies the problem

$$-\Delta u' = F := \varepsilon^{-1}\chi f - 2\nabla\chi \cdot \nabla u - p\chi\varepsilon^{-1}u_x - (q\varepsilon^{-1} + \Delta\chi)u \text{ in } S'',\ u' = 0 \text{ on } \partial S''. \tag{2.3}$$

Using (2.1)

$$\|F\|_0 \le C\varepsilon^{-3/2}\|f\|_0.$$

Hence, from an a priori inequality for the problem (2.3),

$$\|u\|_{2,S'} \le C\|u'\|_{2,S''} \le C\varepsilon^{-3/2}\|f\|_0.$$

Hence, using also (2.1) to estimate the first derivatives of u,

$$\|F\|_{1,S''} \le C\varepsilon^{-5/2}\|f\|_1.$$

Hence

$$\|u\|_{3,S'} \le C\|u'\|_{3,S''} \le C\varepsilon^{-5/2}\|f\|_1.$$

Hence

$$\|F\|_{2,S''} \le C\varepsilon^{-7/2}\|f\|_2.$$

Hence

$$\|u\|_{4,S'} \le C\|u'\|_{4,S''} \le C\varepsilon^{-7/2}\|f\|_2.$$

Continuing in this manner, we get the interior estimates for k an integer. If $s \ne$ integer, the result (2.2) follows from interpolation. ■

We are concerned with the corner singularities in the solution u. We shall use a basic result on corner singularities for the problem

$$-\Delta u = f \text{ in } S,\ u = 0 \text{ on } \Gamma. \tag{2.4}$$

We use the notation $s_j = j\alpha + 1$ for $j = 0, 1, \cdots$.

Theorem 1. *There are linear functionals Λ_j and functions v_j, $j = 1, 2, \cdots$, with the following properties. (i) Λ_j is a bounded linear functional on $H^{s-2}(S)$ for $s > s_j$, but not for $s \le s_j$. (ii) $v_j \in H^s(S) \cap H_0^1(S)$ for $s < s_j$ but not for $s \ge s_j$. Also v_j is smooth everywhere in S except at $r = 0$ and Δv_j is smooth everywhere in $\bar{S}$. (iii) If $s_j < s < s_{j+1}$, $f \in H^{s-2}(S)$, u is a solution of (2.4), and $u \equiv 0$ outside S_1, then*

$$u_j := u - \sum_{i=1}^{j} \Lambda_i(f) v_i \in H^s(S_1), \quad \textit{with } \|u_j\|_{H^s(S_1)} \le C\|f\|_{s-2}.$$

It is useful to have specific formulas for the singular functions and the linear functionals. For the linear functionals we have the formulas

$$\begin{aligned}\Lambda_j(f) = &- \frac{1}{j\pi} \sum_{0\le k+l\le[j\alpha]-2} \frac{A_{k,l}(\omega)}{k!l!(j\alpha-(k+l+2))}[D_{x_1}^k D_{x_2}^l f(0,0)] \\ &+ \frac{1}{j\pi} \int\int_S [f(x,y) - T_{[j\alpha]-2}(x,y)] r^{-j\alpha} \sin j\alpha(\theta-\omega_1)\,dxdy,\ \ j\alpha \ne \text{integer}, \\ \Lambda_j(f) = &\sum_{l=0}^{k-2} C_{k,l}[D_{x_1}^l D_{x_2}^{k-2-l} f(0,0)],\ \ k = j\alpha = \text{integer}.\end{aligned} \tag{2.5}$$

Here, $T_{\lfloor j\alpha\rfloor-2}(x,y)$ denotes the Taylor polynomial of f of degree $\lfloor j\alpha\rfloor - 2$, and the numbers $A_{k,l}(\omega)$ are given by

$$A_{k,l}(\omega) = \int_0^{\omega} \cos^k\phi \sin^l\phi \sin j\alpha\phi\, d\phi.$$

In the case $s \ge 2$, since $s - 2 > j\alpha - 1$, the Sobolev imbedding theorem shows that f has derivatives of order $\lfloor j\alpha\rfloor - 2$, and a Hardy type inequality shows that in the case $j\alpha \ne$ integer, Λ_j is a bounded linear functional on $H^{s-2}(S)$. For $j\alpha =$ integer, the boundedness follows from the Sobolev theorem. In the case $s_1 < s < 2$, which can happen if S is concave, to show that Λ_1 is a bounded linear functional on $H^{s-2}(S) = (H^{2-s}(S))'$ it suffices to show that $r^{-\alpha}\sin\alpha\theta \in H^{2-s}(S)$. This follows from a computation. There are many sets of singular functions that may be used in the theorem. One set is given by the formula

$$v_j(r\cos\theta, r\sin\theta) = \begin{cases} r^{j\alpha}\sin j\alpha(\theta-\omega_1), & j\alpha \ne \text{integer}, \\ (\ln r) r^k \sin k(\theta-\omega_1) + (\theta-\omega_1) r^k \cos k(\theta-\omega_1) & \\ \quad -(-1)^j \omega \csc^k\omega[-x\sin\omega_1 + y\cos\omega_1]^k, & j\alpha = k = \text{integer}\ . \end{cases}$$

One sees by a direct computation that $v_j = 0$ on Γ, Δv_j is smooth, and for each $a > 0$, $v_j \notin H^{s_j}(S_a)$, but $v_j \in H^{s_j-\delta}(S_a)$ for each $\delta > 0$. Also, $\Delta v_j = 0$ if $j\alpha \ne$ integer, and if $j\alpha = k =$ integer, $v_j = O(r^k \ln r)$ near the origin. It can be shown that in the case $j\alpha =$ integer, there is a harmonic singular function but a formula for this singular function does not seem to be known. (Regarding the case $j\alpha = k =$ integer, the function $(\ln r)r^k \sin k(\theta-\omega_1) + (\theta-\omega_1)r^k\cos k(\theta-\omega_1)$ is harmonic; subtraction of the polynomial $(-1)^j\omega\csc^k\omega[-x\sin\omega_1 + y\cos\omega_1]^k$ makes v_k vanish on Γ.)

We denote by $A : f \mapsto u$ the solution operator to the problem (1.1). A is a bounded map from $H^{-1}(S)$ to $H_0^1(S)$. Our approach is to write the problem in the form

$$-\Delta u = F := \varepsilon^{-1}[f - pu_x - qu],\ \text{in } S,\ \ u = 0 \text{ on } \Gamma \tag{2.6}$$

and then apply Theorem 1. In this connection we define a bounded operator $N : H^{-1}(S) \to H^0(S)$ by $Nf = \varepsilon^{-1}[f - p(Af)_x - qAf]$.

The following lemma will prove convenient.

Lemma 2. *Let s and t satisfy $1 < t < s$. Suppose $s_k < s < s_{k+1}$, $s_j < t < s_{j+1}$ for some integers $j \geq 0$ and $k \geq j$. Let $f \in H^{s-2}(S)$, and let u be the solution of (1.1). Suppose u vanishes outside S_1. If $j < k$, then $u \in H^{s'}(S)$ for any $s' < s_{j+1}$. If $j = k$, then $u \in H^s(S)$.*

Proof. Suppose first that $j < k$. Since $u \in H^t(S)$, $|\nabla u| \in H^{t-1}(S)$ so $-\Delta u = F \in H^{t-1}(S)$. There are now two possibilities. If $t - 1 > s_{j+1} - 2$ then $F \in H^{s'-2}(S)$ for any $s' < s_{j+1}$. If $t - 1 \leq s_{j+1} - 2$, then $t - 1 < s - 2$, so $F \in H^{t-1}(S)$. In either case, if $k \geq 1$, the linear functionals $\Lambda_l(F)$ are well defined for $l = 1, \cdots, j$. Since $u \in H^t(S)$ with $t > s_j$ one concludes from Theorem 1 that if $j \geq 1$ $\Lambda_1(F) = \cdots = \Lambda_j(F) = 0$. Hence, again from Theorem 1,

$$u \in H^{s'}(S) \text{ for } s' < s, \text{ if } t - 1 > s_{j+1} - 2,$$
$$u \in H^{t+1}(S) \text{ if } t - 1 \leq s_{j+1} - 2.$$

In the first case the proof is complete. In the second case we repeat the argument with t replaced by $t + 1$, and eventually we conclude that $u \in H^{s'}(S)$ for any $s' < s_{j+1}$.

Now suppose that $j = k$. Thus, $s_k < t < s < s_{k+1}$. We argue as above, but in this case, the regularity in u cannot be increased beyond s, so we obtain $u \in H^s(S)$. ■

We now apply the lemma. Suppose $f \in H^s(S)$ for some large value of s. The solution $u \in H^1(S)$. From Lemma 2 with $t = 1$ we conclude that

$$u \in H^{s'}(S), \ F \in H^{s'-1}(S), \text{ for } s' < s_1$$

Suppose for example that the sector S is convex. This means that $\omega < \pi$, $\alpha > 1$, and $s_1 = \alpha + 1 > 2$. Also one sees that $s_1 + 1 < s_2$. Using Theorem 1 we write

$$u = \Lambda_1(F)v_1 + u_1 \quad \text{with } u_1 \in H^{s'}(S) \text{ for } s' < s_1.$$

Inserting this decomposition of u into the formula for F, we obtain

$$F = f - pu_{1,x} - p_1\Lambda_1(F)v_{1,x} - qu.$$

We see that there is a term in the right side that is in $H^{s'-1}(S)$, namely the term involving $v_{1,x}$. To continue to a higher order expansion, this term must be removed. For this we pick a new "associate" singular function $v_{1,1} \in H^1_0(S)$ such that $-\Delta v_{1,1} = \chi p v_{1,x}$, where χ is a smooth function which vanishes for $r > 1$ and which is $= 1$ in a neighborhood of the origin. Then $u_{1,1} = u - \Lambda_1(F)v_{1,1}$ satisfies

$$-\Delta u_{1,1} = F_{1,1} := f - pu_{1,x} - (1 - \chi)pv_{1,x} - qu \in H^{s'}(S) \text{ for } s' < s_1$$

Thus $F_{1,1}$ has higher regularity and the expansion can be continued using Theorem 1. The function $v_{1,1}$ is a new singular function that arises from the presence of the lower order terms. One has $v_{1,1} \in H^s(S)$ for $s < s_1 + 1$.

Proceeding along the lines sketched above, it would be possible to develop a corner singularity expansion for the solution of (1.1) plus the associated bounds on u. This expansion would require extra "associate" singular functions such as $v_{1,1}$. Instead, an alternate procedure is developed which avoids the extra functions. We will find a function ζ_j such that $L\zeta_j$ is smooth near the origin, $\zeta_j = 0$ on Γ, and $\Lambda_j(-\Delta\zeta_j) = 1$. Using the function ζ_1 for example, we can define u_1 by the decomposition $u = \Lambda_1(F)\zeta_1 + u_1$. Then $\Lambda_1(-\Delta u_1) = 0$, but now $Lu_1 = f - \Lambda_1(F)L\zeta_1$ has the same regularity as f, so the expansion can be continued without recourse to the extra singular function $v_{1,1}$.

To find such functions ζ_j, let $d = 1/(2\varepsilon)$. We start with the formula $L(e^{dpx}z) = \frac{1}{2}d[-4\varepsilon^2\Delta z + \kappa^2 z]e^{dpx}$, where $\kappa^2 = p^2 + 4\varepsilon q$. We will use stretched variables $\xi = dx$, $\eta = dy$, and we will also write $\rho = (\xi^2+\eta^2)^{1/2} = dr$. Setting $z(x, y) = z_1(\xi, \eta)$, we have

$$L(e^{dpx}z) = \frac{1}{2}d[-\Delta_{(\xi,\eta)}z_1 + \kappa^2 z_1]e^{p\xi}.$$

If z_1 is a function of the form $z_1(\xi, \eta) = R(\rho)\sin j\alpha(\theta - \omega_1)$ we obtain

$$L(e^{dpx}z) = -\frac{1}{2}de^{p\xi}(M_jR)\sin j\alpha(\theta - \omega_1),$$

where the differential operator M_j is defined by

$$M_jR = R'' + \rho^{-1}R' - (j^2\alpha^2 + \kappa^2\rho^2)\rho^{-2}R.$$

Note that $R(\rho) = I_{j\alpha}(\kappa\rho)$ or $R(\rho) = K_{j\alpha}(\kappa\rho)$ solves the equation $M_jR = 0$. Concerning the modified Bessel functions $I_{j\alpha}$ and $K_{j\alpha}$ recall that for $j > 0$

$$\begin{aligned} &I_{j\alpha}(t) \sim A_jt^{j\alpha} + O(t^{j\alpha+2}),\ K_{j\alpha}(t) \sim B_jt^{-j\alpha} + O(t^{-j\alpha+2}), \quad \text{for } t \text{ small},\\ &I_{j\alpha}(t) \sim t^{-1/2}e^{t},\ K_{j\alpha}(t) \sim t^{-1/2}e^{-t}, \quad \text{for } t \text{ large.} \end{aligned} \tag{2.7}$$

The numbers A_j, B_j are given in [1, (9.6.7), (9.6.9)]. The derivatives obey the relations obtained by differentiating (2.7).

We now finish the definition of ζ_j. Let ψ_j be a smooth function on $(0, \infty)$ such that $\psi_j > 0$ in $(1, 2)$ and $\psi_j = 0$ outside $(1, 2)$. In addition we require that ψ_j is normalized so that $\Lambda_j(-\Delta\psi_j) = 1$. In the case $j\alpha \neq$ integer, Lemma 3 below shows that this is acomplished with the normalization

$$\int\int_S \psi_j(\rho)K_{j\alpha}(\kappa\rho)\sin^2 j\alpha(\theta - \omega_1)d\xi d\eta = 2j\pi\kappa^{-j\alpha}d^{1-j\alpha}B_j. \tag{2.8}$$

We do not discuss the normalization in the case $j\alpha =$ integer. Having selected ψ_j we define $R_j(\rho)$ to be the solution to the two point boundary value problem

$$M_jR_j = -\psi_j(\rho),\ R_j(0) = 0,\ R_j(r) \to 0 \text{ as } \rho \to \infty, \tag{2.9}$$

and we define our singular functions ζ_j by

$$\begin{aligned} \zeta_j(x, y) &= e^{p\xi}R_j(\rho)\sin j\alpha(\theta - \omega_1), \quad \text{if } j\alpha \neq \text{integer},\\ \zeta_j(x, y) &= e^{p\xi}(\ln\rho)R_j(\rho)\sin k(\theta - \omega_1) + ke^{p\xi}R_j(\rho)(\theta - \omega_1)\cos k(\theta - \omega_1)\\ &\quad - (-1)^jk\omega e^{p\xi}R_j\big(\csc\omega[-\xi\sin\omega_1 + \eta\cos\omega_1]\big), \quad \text{if } j\alpha = k = \text{integer}. \end{aligned} \tag{2.10}$$

Some properties of ζ_j are given in Lemma 4. The following lemma will be used several times.

Lemma 3. *Let $j \geq 1$ with $j\alpha \neq$ integer, let $v \in H^{s'}(S) \cap H^1_0(S)$ for any $s' < s_j$, and suppose $Lv \in H^{s-2}(S)$ for some $s_j < s < s_{j+1}$. Suppose in addition that Lv and its derivatives of order $\leq \lfloor j\alpha \rfloor - 2$ vanish at $r = 0$. Then*

$$\Lambda_j(-\Delta v) = \frac{C_j}{j\pi}\varepsilon^{-j\alpha}\int\int_S (Lv)(x, y)e^{-p\xi}K_{j\alpha}(\kappa\rho)\sin j\alpha(\theta - \omega_1)dxdy,\ j\alpha \neq \textit{integer}, \tag{2.11}$$

where $C_j = \kappa^{j\alpha} 2^{-j\alpha} B_j^{-1}$.

Proof. From the hypotheses, Lv is continuously differentiable of order $\lfloor j\alpha \rfloor - 2$, so the vanishing condition on Lv and its derivatives has meaning. Since $-\Delta v = Lv - pv_x - qv$ and $v_x \in H^{s'-2}(S)$ for any $s' < s_j + 1$, $-\Delta v \in H^{s''-2}(S)$ for some $s'' > s_j$. Hence $\Lambda_j(-\Delta v)$ is well-defined. To calculate $\Lambda_j(-\Delta v)$, we use the formula (2.5). This formula requires the Taylor polynomial of degree $\lfloor j\alpha \rfloor - 2$ of $-\Delta v$. By hypothesis, the Taylor polynomial of Lv is zero. Since $v \in H^{s'}(S)$ for any $s' < s_j = j\alpha + 1$ and $j\alpha \neq$ integer, v has continuous derivatives of order $\leq \lfloor j\alpha \rfloor$. We now sketch an inductive argument to show that all the derivatives of order $\leq \lfloor j\alpha \rfloor$ of v vanish at the origin. Since v vanishes on the two sides of S, $v = v_x = v_y = 0$ at the origin. Let $k \leq \lfloor j\alpha \rfloor$ and suppose that all derivatives of order $< k$ of v vanish at the origin. Using the fact that the derivatives of order $k-2$ of Lv vanish at the origin, we obtain $k-2$ homogeneous linear equations for the k–th order derivatives of v at the origin. Since v vanishes on the two sides of S we obtain 2 homogeneous linear equations for the k–th order derivatives of v at the origin. It may be seen that these $k+1$ equations are linearly dependent. Therefore all the derivatives of v of order $\leq \lfloor j\alpha \rfloor$ vanish at the origin. As a consequence we have

$$|v(x,y)| + r|Dv(x,y)| \leq Cr^{\lfloor j\alpha \rfloor} \quad \text{near the origin.} \tag{2.12}$$

Hence the Taylor polynomial of degree $\lfloor j\alpha \rfloor - 2$ of $-\Delta v$ is zero and we have

$$\Lambda_j(-\Delta v) = \frac{1}{j\pi} \int\int_S (-\Delta v) r^{-j\alpha} \sin j\alpha(\theta - \omega_1) dxdy. \tag{2.13}$$

To prove (2.11) we will use the the adjoint operator L^*, defined by $L^* u := -\varepsilon \Delta u - pu_x + qu$, and we will use the function $\tilde{K}$ defined by $\tilde{K}(x,y) = e^{-p\xi} K_{j\alpha}(\kappa\rho)$. Note that $\varepsilon\Lambda_j(-\Delta v) = \Lambda_j(Lv) - \Lambda_j(pv_x + qv)$. To show (2.11), using (2.13) we must show that

$$\int\int_S Lv\{r^{-j\alpha} \sin j\alpha(\theta-\omega_1) - C_j \varepsilon^{-j\alpha} \tilde{K} \sin j\alpha(\theta-\omega_1)\} dxdy = \int\int_S r^{-j\alpha} \sin j\alpha(\theta-\omega_1)(pv_x + qv) dxdy. \tag{2.14}$$

One may check that the integrals on both sides of (2.14) are convergent. To show the equality in (2.14) careful attention must be paid to the singularity at $r = 0$. Letting $S_\delta^c = S \setminus S_\delta$, we write

$$\begin{aligned}
&\int\int_S Lv\{r^{-j\alpha} \sin j\alpha(\theta - \omega_1) - C_j \varepsilon^{-j\alpha} \tilde{K} \sin j\alpha(\theta - \omega_1)\} dxdy \\
&\quad = \lim_{\delta\to 0} \int\int_{S_\delta^c} Lv\{r^{-j\alpha} \sin j\alpha(\theta - \omega_1) - C_j \varepsilon^{-j\alpha} \tilde{K} \sin j\alpha(\theta - \omega_1)\} dxdy \\
&\quad = \lim_{\delta\to 0} \int\int_{S_\delta^c} v\{L^*(r^{-j\alpha} \sin j\alpha(\theta - \omega_1)) - C_j \varepsilon^{-j\alpha} L^*(\tilde{K} \sin j\alpha(\theta - \omega_1))\}) dxdy \\
&\qquad + \lim_{\delta\to 0} \int_{r=\delta} \varepsilon v_r \{r^{-j\alpha} \sin j\alpha(\theta - \omega_1) - C_j \varepsilon^{-j\alpha} \tilde{K} \sin j\alpha(\theta - \omega_1)\} \delta d\theta \\
&\qquad - \lim_{\delta\to 0} \int_{r=\delta} \varepsilon v\{-j\alpha r^{-j\alpha-1} \sin j\alpha(\theta - \omega_1) - C_j \varepsilon^{-j\alpha} \tilde{K}_r \sin j\alpha(\theta - \omega_1)\} \delta d\theta \\
&\qquad - \lim_{\delta\to 0} \int_{r=\delta} p(\cos(\theta - \omega_1)) v\{r^{-j\alpha} \sin j\alpha(\theta - \omega_1) - C_j \varepsilon^{-j\alpha} \tilde{K} \sin j\alpha(\theta - \omega_1)\} \delta d\theta.
\end{aligned}$$

A calculation shows that $L^*(\tilde{K} \sin j\alpha(\theta - \omega_1)) = 0$. We now show that the integrals over $r = \delta$ tend to zero as $\delta \to 0$. This calculation utilizes the formula for C_j. To verify that

$$\lim_{\delta\to 0} \int_{r=\delta} v_r \{r^{-j\alpha} \sin j\alpha(\theta - \omega_1) - C_j \varepsilon^{-j\alpha} \tilde{K} \sin j\alpha(\theta - \omega_1)\} \delta d\theta = 0 \tag{2.15}$$

we use (2.7) and the definition of C_j to write

$$|v_r(r^{-j\alpha} - C_j\varepsilon^{-j\alpha}\tilde{K})| \leq Cr^{\lfloor j\alpha \rfloor - 1}\big|r^{-j\alpha} - C_j\varepsilon^{-j\alpha}B_j(\kappa\rho)^{-j\alpha} + O(r^{-j\alpha+1})\big| = O(r^{\lfloor j\alpha \rfloor - j\alpha}).$$

Setting $r = \delta$ and using this in the integral, we obtain (2.15). The other integral is handled in the same way. We therefore get

$$\begin{aligned}\int\!\!\int_S Lv\{r^{-j\alpha}\sin j\alpha(\theta-\omega_1) - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha(\theta-\omega_1)\}dxdy& \\ = \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c} vL^*(r^{-j\alpha}\sin j\alpha(\theta-\omega_1))dxdy.&\end{aligned}$$

Since $r^{-j\alpha}\sin j\alpha(\theta-\omega_1)$ is a harmonic function, $L^*(r^{-j\alpha}\sin j\alpha(\theta-\omega_1)) = -p(r^{-j\alpha}\sin j\alpha(\theta-\omega_1))_x + qr^{-j\alpha}\sin j\alpha(\theta-\omega_1)$. Hence

$$\begin{aligned}&\int\!\!\int_S Lv\{r^{-j\alpha}\sin j\alpha(\theta-\omega_1) - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha(\theta-\omega_1)\}dxdy\\ &\quad = \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c} v\{-p(r^{-j\alpha}\sin j\alpha(\theta-\omega_1))_x + qr^{-j\alpha}\sin j\alpha(\theta-\omega_1))\}dxdy\\ &\quad = \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c}(pv_x+qv)(r^{-j\alpha}\sin j\alpha(\theta-\omega_1))dxdy\\ &\qquad + \lim_{\delta\to 0}\int_{r=\delta}p(\cos(\theta-\omega_1))vr^{-j\alpha}\sin j\alpha(\theta-\omega_1))\delta d\theta.\end{aligned}$$

Again it is seen that the integral over $r = \delta$ tends to zero as $\delta \to 0$. This completes the proof of (2.14), and hence of (2.11). ■

Lemma 4. *The function ζ_j vanishes on the two sides of S. One has $\zeta_j \in H^s(S)$ if and only if $s < s_j$. Also, $L\zeta_j$ is smooth in $\bar{S}$ and*

$$\begin{aligned}&L\zeta_j = \frac{1}{2}de^{p\xi}\psi_j(\rho)\sin j\alpha(\theta-\omega_1)\quad \textit{if } j\alpha \neq \textit{integer},\\ &L\zeta_j = \frac{1}{2}de^{p\xi}(\ln\rho)\psi_j(\rho)\sin j\alpha(\theta-\omega_1) + \frac{1}{2}dke^{p\xi}(\theta-\omega_1)\psi_j(\rho)\cos j\alpha(\theta-\omega_1)\\ &\quad + de^{p\xi}[k\rho^{-2}R_j(\rho) - \rho^{-1}R_j'(\rho)]\sin j\alpha(\theta-\omega_1)\\ &\quad - (-1)^j k\omega e^{p\xi}\big(-\csc\omega^2 R_j''(\csc\omega[-\xi\sin\omega_1+\eta\cos\omega_1]) + \kappa^2 R_j(\csc\omega[-\xi\sin\omega_1+\eta\cos\omega_1])\big)\\ &\qquad\qquad \textit{if } j\alpha = k = \textit{integer},\end{aligned}\tag{2.16}$$

$$|D^m\zeta_j(x,y)| \leq C\varepsilon^{-j\alpha-m}r^{j\alpha-m}\quad \textit{for } r \leq C_1\varepsilon,\tag{2.17}$$

$$|D^m\zeta_j(x,y)| \leq C\varepsilon^{-m}D^m_{\xi\eta}\left(\rho^{-1}\exp\left\{-\left((q+\tfrac{1}{4}p^2\rho)^{1/2} - \frac{1}{2}p\xi\right)\right\}\right)\quad \textit{for } r \geq C_1\varepsilon,\tag{2.18}$$

$$\Lambda_j(-\Delta\zeta_j) = 1.\tag{2.19}$$

Proof. The vanishing of ζ_j on the sides of S follows from the definition of ζ_j, as does the formula (2.16). Since $M_jR_j = 0$ in $(0,1)$, in the case $j\alpha \neq$ integer, $L\zeta_j = 0$ near the origin so Lz_j is smooth in this case. Also since $M_jR_j = 0$ in $(0,1)$, $R_j(\rho)$ is a linear combination of $I_{j\alpha}(\kappa\rho)$ and $K_{j\alpha}(\kappa\rho)$ in $(0,1)$, and the boundary condition in (2.9) implies that $R_j(\rho) = CI_{j\alpha}(\kappa\rho)$ there. Equation (2.17) then follows from (2.7). Furthermore, if $j\alpha = k =$ an integer, $I_{j\alpha}(\rho) = I_k(\rho)$ is given by a convergent power series with leading term ρ^k, so $L\zeta_j$ is smooth in this case, and (2.17) holds in this case. Since $M_jR_j = 0$ for $\rho > 2$, $R_j(\rho)$ is a linear combination of $I_{j\alpha}(\kappa\rho)$ and $K_{j\alpha}(\kappa\rho)$ for $\rho > 2$, and the boundary condition in (2.9) implies that $R_j(\rho) = CK_{j\alpha}(\kappa\rho)$ there, and (2.18) follows from (2.7). Using (2.17) and (2.18), we see that $\zeta_j \in H^s(S)$ if and only if $s < s_j$. Finally, (2.19) follows from (2.16) and Lemma 3. ■

We remark that in the case $j\alpha \neq$ integer $L\zeta_j$ satisfies the inequality

$$\|L\zeta_j\|_k \leq C\varepsilon^{2j-k-1}.$$

Letting D^k denote any derivative of order k, this follows from the inequalities

$$\begin{aligned}\|D^k_{xy}L\zeta_j\|_0 &= \left(\int\int_S |D^k_{xy}L\zeta_j|^2 dxdy\right)^{1/2} \\ &= d^{k-1}\left(\int\int_S |D^k_{\xi\eta}L\zeta_j|^2 d\xi d\eta\right)^{1/2} \\ &= \frac{1}{2}d^k\left(\int\int_S |D^k_{\xi\eta}\big(e^{p\xi}\psi_j(\rho)\sin j\alpha(\theta-\omega_1)\big)|^2 d\xi d\eta\right)^{1/2}\end{aligned}$$

In the last inequality the normalization (2.8) has been used. It would be of interest to construct singular functions such that $\|L\zeta_j\|_k$ is bounded independent of ε, or at least bounded independent of ε on regions of size ε^{-1}.

The next result gives the corner singularity expansion for the problem (1.1).

Theorem 2. *There are linear functionals $\Lambda_{L,j}$ and functions ζ_j, $j = 1, 2, \cdots$, with the following properties. (i) $\Lambda_{L,j}$ is a bounded linear functional on $H^{s-2}(S)$ for $s > s_j$, but not for $s \leq s_j$. (ii) $\zeta_j \in H^s(S) \cap H^1_0(S)$ for $s < s_j$ but not for $s \geq s_j$. Also ζ_j is smooth everywhere except at the origin and $L\zeta_j$ is smooth everywhere in $\bar{S}$. (iii) If $j > 0$, $s_j < s < s_{j+1}$, $f \in H^{s-2}(S)$ and $u = Af$, then*

$$u_j := u - \sum_{k=1}^{j} \Lambda_{L,k}(f)\zeta_k \in H^s(S). \tag{2.20}$$

The linear functionals $\Lambda_{L,k}$ and the remainder u_j satisfy

$$\sum_{k=1}^{j} \varepsilon^{\sigma_k}|\Lambda_{L,k}(f)| + \varepsilon^{\sigma(s)}\|u_j\|_s \leq C\|f\|_{s-2} \tag{2.21}$$

where the exponents σ_k depend only on k and ω, and the exponent $\sigma(s)$ depends only on s and the angle ω. If $j = 0$, the sum is replaced by 0 and $u = u_0 \in H^s(S)$ and satisfies (2.21).

Proof. The definition of ζ_j is given in (2.9) and the assertions (ii) follow from Lemma 4. The rest of the proof is given in a series of steps. In the proof, χ denotes a smooth function which is $\equiv 1$ in a neighborhood of the origin and $\equiv 0$ for $r \geq 1$.

(a). Suppose $1 < s < s_1$. Since $u \in H^1(S)$, $\chi u \in H^1(S)$ and $L(\chi u) \in H^{\min(s-2,0)}(S)$. Hence $NL(\chi u) \in H^{\min(s-2,0)}(S)$. Lemma 2 implies that $\chi u \in H^s(S)$. From Lemma 1, $(1-\chi)u \in H^s(S)$. This gives the decomposition (2.20) in the case $j = 0$.

(b). Suppose that $s_1 < s < s_2$. From (a), $u \in H^{s'}(S)$ for any $s' < s_1$. Hence $\chi u \in H^{s'}(S)$. Hence $F = NL(\chi u) \in H^{s'-2}(S)$ for any $s' < s_1 + 1$. Pick a number $s' > s_1$ with $s' < \min(s, s_1 + 1, s_2)$. Then $\Lambda_1(F)$ is well-defined. We define

$$\Lambda_{L,1}(f) = \Lambda_1(F) = \Lambda_1(-\Delta(\chi u)). \tag{2.22}$$

Let $u_1^* = \chi u - \Lambda_{L,1}(f)\zeta_1$. Then $\Lambda_{L,1}(Lu_1^*) = \Lambda_1(-\Delta u_1^*) = 0$ so from Theorem 1, $u_1^* \in H^{s'}(S)$. Also $Lu_1^* \in H^{s-2}(S)$. Applying Lemma 2, $u_1^* \in H^s(S)$. From Lemma 1, $(1-\chi)u \in H^s(S)$. Setting $u_1 = u_1^* + (1-\chi)u$, we have $u = \Lambda_{L,1}(f)\zeta_1 + u_1$ with $u_1 \in H^s(S)$. This gives the decomposition (2.20) in the case $j = 1$.

(c) We use an argument by induction and a recursive definition of the linear functionals. Suppose the linear functionals $\Lambda_{L,i}$ have been defined for $i < j$. Let $s_j < s < s_{j+1}$. Using induction, we write $u = \sum_{k=1}^{j-1} \Lambda_{L,k}(f)\zeta_k + u_{j-1}$ with $u_{j-1} \in H^{s'}(S)$ for any $s' < s_{j-1}$. Hence $\chi u_{j-1} \in H^{s'}(S)$. Hence $F = NL(\chi u_{j-1}) \in H^{s'-2}(S)$ for any $s' < s_{j-1} + 1$. Pick a number $s' > s_{j-1}$ with $s' < \min(s, s_{j-1} + 1, s_2)$. Then $\Lambda_j(F)$ is well-defined. We define

$$\Lambda_{L,j}(f) = \Lambda_j(F) = \Lambda_j(-\Delta(\chi u_{j-1})). \tag{2.23}$$

Let $u_j^* = \chi u_{j-1} - \Lambda_{L,j}(f)\zeta_j$. Then $\Lambda_{L,j}(Lu_j^*) = \Lambda_j(-\Delta u_j^*) = 0$ so from Theorem 1, $u_j^* \in H^{s'}(S)$. Also $Lu_j^* \in H^{s-2}(S)$. Applying Lemma 2, $u_j^* \in H^s(S)$. Again we have $(1-\chi)u_{j-1} \in H^s(S)$. Setting $u_j = u_j^* + (1-\chi)u_{j-1}$, we have $u = \Lambda_{L,j}(f)\zeta_j + u_j$ with $u_j \in H^s(S)$. This gives the decomposition (2.20) in the general case.

(d) The form of the constants in (2.21), and a value for σ_k and $\sigma(s)$, come from examining the remainder in each step of the recursive argument. ■

In the proof of Theorem 2, the linear functionals $\Lambda_{L,j}$ are defined recursively by (2.22) and (2.23). We are now concerned with formulas for calculating these linear functionals. In these formulas we will assume that $j\alpha \neq$ integer.

Lemma 5. *Suppose $j\alpha \neq$ integer. Let $f \in H^{s-2}(S)$ with $s > s_j$. Suppose also that $s > 3/2$ and $j\alpha > \frac{1}{2}$. Then*

$$\Lambda_{L,1}(f) = \Lambda_1(\{-\Delta u\}),$$
$$\Lambda_{L,j}(f) = \Lambda_j(\{-\Delta u_{j-1}\}).\ j \geq 2$$

Proof. We will show that

$$\Lambda_j(-\Delta((1-\chi)u_{j-1})) = 0. \tag{2.24}$$

Adding (2.23) and (2.24) then gives the result. To show (2.24), write $w = (1-\chi)u_{j-1}$ Since w vanishes in a neighborhood of the origin, its Taylor polynomial is 0. Letting Γ_R denote the curved portion of the boundary of the truncated sector S_R, and using Green's theorem and (2.5),

$$\begin{aligned} |\Lambda_j(-\Delta w)| &= \frac{1}{j\pi}\left|\int\int_S (\Delta w) r^{-j\alpha} \sin j\alpha(\theta - \omega_1)\,dxdy\right| \\ &= \frac{1}{j\pi}\left|\lim_{R\to\infty} \int\int_{S_R} (\Delta w) r^{-j\alpha} \sin j\alpha(\theta - \omega_1)\,dxdy\right| \\ &\leq \frac{1}{j\pi} \lim_{R\to\infty} \left\{ R^{-j\alpha} \int_{\Gamma_R} |w_r| R\,d\theta + R^{-j\alpha-1} \int_{\Gamma_R} |w| R\,d\theta \right\}. \end{aligned}$$

Since $s > 3/2$, $w \in H^{s'}(S)$ for some $s' > 3/2$. Hence $w_r \in L_2(\Gamma_R)$ and $\|w_r\|_{0,\Gamma_R} \leq C\|w\|_{s'}$. Schwarz's inequality shows that the last two terms are bounded by $CR^{-j\alpha+\frac{1}{2}} = o(1)$ as $R \to \infty$. ■

The next lemma gives a formula for $\Lambda_{L,j}$ analogous to the formulas (2.5) for Λ_j, provided f vanishes to as high an order as possible at the origin.

Lemma 6. *Suppose $j\alpha \neq$ integer. Let $s_j < s < s_{j+1}$ for some $j \geq 1$ and let $f \in H^{s-2}(S)$, so f is continuously differentiable of order $\lfloor j\alpha \rfloor - 2$. Suppose in addition that f and its derivatives of order $\leq \lfloor j\alpha \rfloor - 2$ vanish at $r = 0$. Then*

$$\Lambda_{L,j}(f) = \frac{C_j}{j\pi}\varepsilon^{-j\alpha-1}\int\int_S f(x,y)e^{-p\xi}K_{j\alpha}(\kappa\rho)\sin j\alpha(\theta-\omega_1)dxdy. \tag{2.25}$$

Proof. Let $u_0 = u$. Then from Lemma 5, $\Lambda_{L,j}(f) = \Lambda_j\{-\Delta u_{j-1}\}$. Let $f_{j-1} = Lu_{j-1}$. From (2.20) and the fact that $L\zeta_i$ vanishes in a neighborhood of the origin, f_{j-1} satisfies the same vanishing condition as f. Hence from Lemma 3

$$\begin{aligned}\Lambda_{L,j}(f_{j-1}) &= \Lambda_j(-\Delta u_{j-1}) \\ &= \frac{1}{j\pi\varepsilon}C_j\varepsilon^{-j\alpha}\int\int_S f_{j-1}(x)e^{-p\xi}K_{j\alpha}(\kappa\rho)\sin j\alpha(\theta-\omega_1)dxdy.\end{aligned}$$

This proves the result in the case $j = 1$. For $j > 1$ we use the formula (2.16) and the orthogonality relation

$$\int_{\omega_1}^{\omega_2}\sin i\alpha(\theta-\omega_1)\sin j\alpha(\theta-\omega_2)d\theta = 0 \text{ for } i \neq j$$

to obtain $\Lambda_{L,j}(L\zeta_i) = 0$ for $i \neq j$. Using (2.20), the lemma is proved. ■

We now consider bounds for the linear functionals and their dependence on ε.

Lemma 7. *Suppose $j\alpha \neq$ integer. Let f satisfy the hypotheses of Lemma 6. Then*

$$|\Lambda_{L,j}(f)| \leq C\varepsilon^{s-s_j-1}\|f\|_{s-2}.$$

Proof. Since $f \in H^{s-2}(S)$ and vanishes to order $\lfloor j\alpha - 2\rfloor$ at the origin, using (2.7) and the condition $s > s_j$,

$$\begin{aligned}|\Lambda_{L,j}(f)| &\leq C\varepsilon^{-j\alpha-1}\int\int_S |f||K_{j\alpha}(\kappa\rho)|dxdy \\ &\leq C\varepsilon^{-j\alpha-1}\int\int_S \frac{|f|}{r^{s-2}}|K_{j\alpha}(\kappa\rho)|r^{s-2}dxdy \\ &\leq C\varepsilon^{-j\alpha-1}\|f\|_{s-2}\left\{\int\int_S |K_{j\alpha}(\kappa\rho)|^2 r^{2s-4}dxdy\right\}^{1/2} \\ &\leq C\varepsilon^{s-1-j\alpha-1}\|f\|_{s-2}\left\{\int\int_S |K_{j\alpha}(\kappa\rho)|^2\rho^{2s-4}d\xi d\eta\right\}^{1/2} \\ &\leq C\varepsilon^{s-s_j-1}\|f\|_{s-2}.\end{aligned}$$

In these inequalities we have also used a Hardy-type inequality. ■

Using this we obtain a bound on the residual that is better than the bound on the remainder given by Theorem 2.

Lemma 8. *Suppose $j\alpha \neq$ integer. Let f satisfy the hypotheses of Lemma 6. Then the remainder u_j in the decomposition of Theorem 2 satisfies*

$$\|Lu_j\|_{s-2} \leq C\|f\|_{s-2}.$$

Proof. Let $\bar\psi_j = \varepsilon^{-j\alpha}\psi_j$. Using (2.19) it is seen that $\bar\psi_j$ does not depend on ε, it is a function with support in $(1,2)$, and it satisfies

$$\int\int_S \bar\psi_j(\rho)K_{j\alpha}(\kappa\rho)\sin^2 j\alpha(\theta-\omega_1)d\xi d\eta = -j\pi C_j^{-1}.$$

In terms of $\bar{\psi}_j$ we have

$$L\zeta_j = \tfrac{1}{4}\varepsilon^{j\alpha-1}e^{p\xi}\bar{\psi}_j(\rho)\sin j\alpha(\theta-\omega_1).$$

Since $D^k_{x,y} = (2\varepsilon)^{-k}D^k_{\xi,\eta}$ we have

$$\|L\zeta_j\|_s \leq C\varepsilon^{j\alpha-s}.$$

Hence

$$\begin{aligned}\|Lu_j\|_{s-2} &\leq C\|f\|_{s-2} + \sum_1^j |\Lambda_{L,l}(f)|\|L\zeta_j\|_{s-2}\\ &\leq C\|f\|_{s-2} + C\sum_1^j \varepsilon^{s-s_j-1}\|f\|_{s-2}\cdot\varepsilon^{j\alpha-(s-2)}\\ &\leq C\|f\|_{s-2}.\end{aligned}$$

■

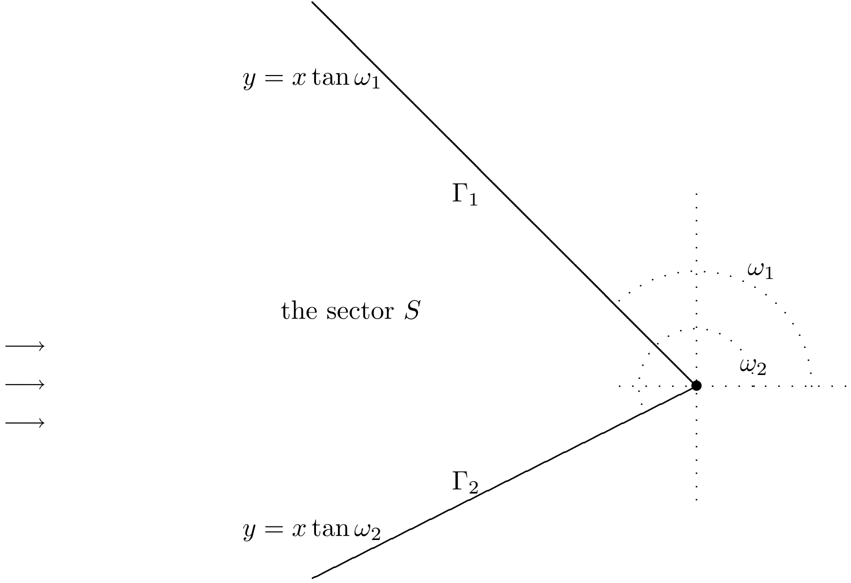
One does not, in general, expect derivative bounds for the remainder u_j that are independent of ε. The reason is that although the remainder u_j has no corner singularity, it still may have boundary layer behavior. In the case of an *incoming sector*, there are no boundary layers and one can imagine that the derivatives of u_j are bounded independent of ε, provided the data satisfies the vanishing conditions.

3. An outgoing sector

In this section we consider the problem (1.1) in an "outgoing sector", situated as in the figure. An asymptotic expansion of the solution which involves the "reduced" differential operator, and an asymptotic boundary layer expansion which uses a stretched variable near the boundary, is constructed. When these expansions are subtracted from the solution, there is obtained a boundary value problem with small right hand side and exponentially decaying boundary data. The boundary data are treated using an energy argument that also involves some results of Section 2. There remains a boundary value problem for the remainder with zero boundary data and small right hand side. The results of Section 2 are used to obtain bounds for the remainder and its derivatives. The final result, stated in Theorem 3, consists of pointwise bounds on the derivatives of the solution that exhibit the boundary layer behavior and the corner singularity. These pointwise bounds make rather heavy demands on the differentiability of the data f. For convenience, we assume in this section that $f \in H^s(S)$ for any $s > 0$. Unfortunately, the analysis requires a convex sector. It is thus assumed that

$$\omega = \omega_2 - \omega_1 < \pi \text{ and } \; 0 < \omega_1 < \pi < \omega_2 < 2\pi.$$

It seems likely that these restrictions are not necessary, but are imposed by our proof.



Consider the sequence of reduced solutions associated with the problem (1.1). Thus, let M denote the reduced differential operator, defined by $Mv = pv_x + qv$, and define functions $v_0, v_1, \cdots$, by the problems

$$\begin{aligned} &Mv_0 = f \text{ in } S,\; v_0(x,y) \to 0 \text{ as } x \to -\infty, \\ &Mv_k = -\Delta v_{k-1} \text{ in } S,\;\; v_k(x,y) \to 0 \text{ as } x \to -\infty,\; k = 1, 2 \cdots. \end{aligned} \tag{3.1}$$

Regarding the problems (3.1) we cite without proof the following simple lemma.

Lemma 9. *The problem*

$$MV = F \text{ in } S,\; V(x,y) \to 0 \text{ as } x \to -\infty$$

has the solution

$$V(x,y) = p^{-1} \int_{-\infty}^{x} e^{-qp^{-1}(x-s)} F(s,y)ds.$$

If $F \in H^s(S)$ for some $s \geq 0$, then $V \in H^s(S)$ and $\|V\|_s \leq C(s)\|F\|_s$.

Applying Lemma 9 with an inductive argument, one finds from (3.1) that if $f \in H^{2k+s}(S)$, then $v_k \in H^s(S)$ and

$$\|v_k\|_s \le C\|f\|_{2k+s}. \tag{3.2}$$

Set $V_K = \sum_0^K \varepsilon^k v_k$. Using (3.1), one obtains $LV_K = f - \varepsilon^{K+1}\Delta v_K$ so setting $u_1 = u - V_K$, u_1 satisfies the problem

$$Lu_1 = f_1 \text{ in } S,\ u_1 = g_{1,l} \text{ on } \Gamma_l,\ l = 1, 2, \tag{3.3}$$

where we have set

$$f_1 = \varepsilon^{K+1}\Delta v_K,\ g_{1,l}(r) = -V_K(r\cos\omega_l, r\sin\omega_l),\ l = 1, 2. \tag{3.4}$$

One has $g_{1,1}(0) = g_{1,2}(0)$. The integer K will be chosen below.

Regarding the data of the problem (3.4), we note that while $f_1 = O(\varepsilon^{K+1})$, the boundary values $g_{1,1}$ and $g_{1,2}$ do not vanish as $\varepsilon \to 0$. We next construct some boundary layer functions to deal with this behavior. We use the distances d_l along Γ_l and e_l perpendicular to Γ_l, with the signs arranged so that these distances are positive in S. It at this step that the convexity of S is used, as the positivity of e_1 and e_2 requires a convex sector. These quantities are given by the formulas

$$\begin{aligned} d_1 &= x\cos\omega_1 + y\sin\omega_1,\ e_1 = -x\sin\omega_1 + y\cos\omega_1, \\ d_2 &= x\cos\omega_2 + y\sin\omega_2,\ e_2 = x\sin\omega_2 - y\cos\omega_2. \end{aligned} \tag{3.5}$$

We will also use the stretched variables $E_1 = e_1/\varepsilon$ and $E_2 = e_2/\varepsilon$. To represent the boundary layer along Γ_1 we will use a function $w_1(E_1, e_2)$. For derivatives of w_1 we have

$$\frac{\partial}{\partial x}w_1(E_1, e_2) = -\varepsilon^{-1}w_{1,E_1}\sin\omega_1 + w_{1,e_2}\sin\omega_2,\ \frac{\partial}{\partial y}w_1(E_1, e_2) = \varepsilon^{-1}w_{1,E_1}\cos\omega_1 - w_{1,e_2}\cos\omega_2,$$

$$\frac{\partial^2}{\partial x^2}w_1(E_1, e_2) = \varepsilon^{-2}w_{1,E_1E_1}\sin^2\omega_1 - 2\varepsilon^{-1}w_{1,E_1e_2}\sin\omega_1\sin\omega_2 + w_{1,e_2e_2}\sin^2\omega_2,$$

$$\frac{\partial^2}{\partial y^2}w_1(E_1, e_2) = \varepsilon^{-2}w_{1,E_1E_1}\cos^2\omega_1 - 2\varepsilon^{-1}w_{1,E_1e_2}\cos\omega_1\cos\omega_2 + w_{1,e_2e_2}\cos^2\omega_2,$$

$$\begin{aligned} Lw_1 = {} & -\varepsilon^{-1}\left[w_{1,E_1E_1} + pw_{1,E_1}\sin\omega_1\right] \\ & + 2w_{1,E_1e_2}\cos\omega + pw_{1,e_2}\sin\omega_2 + qw_1 - \varepsilon w_{1,e_2e_2}. \end{aligned}$$

Using these formulas, we define a sequence of functions $w_{1,j}(E_1, e_2)$ by writing $L(\sum_0 \varepsilon^j w_{1,j}(E_1, e_2)) = 0$ and equating coefficients of ε. In this way we arrive at the sequence of two point boundary value problems

$$w_{1,0,E_1E_1} + pw_{1,0,E_1}\sin\omega_1 = 0,\ w_{1,0}(0, e_2) = g_{1,1}\big(e_2\csc\omega\big), w_{1,0}(E_1, e_2) \to 0 \text{ as } E_1 \to \infty.$$

$$\begin{aligned} w_{1,1,E_1E_1} + pw_{1,1,E_1}\sin\omega_1 &= 2w_{1,0,E_1e_2}\cos\omega + pw_{1,0,e_2} + qw_{1,0}, \\ w_{1,1}(0, e_2) &= 0, w_{1,1}(E_1, e_2) \to 0 \text{ as } E_1 \to \infty, \end{aligned}$$

$$\begin{aligned} w_{1,j,E_1E_1} + pw_{1,j,E_1}\sin\omega_1 &= 2w_{1,j-1,E_1e_2}\cos\omega + pw_{1,j-1,e_2} + qw_{1,j-1} - w_{1,j-2,e_2e_2}, \\ w_{1,j}(0, e_2) &= 0, w_{1,j}(E_1, e_2) \to 0 \text{ as } E_1 \to \infty,\ \text{for } j > 1. \end{aligned}$$

Setting $W_{1,J} = \sum_0^J w_{1,j}$ we have

$$LW_{1,J} = \varepsilon^J (2w_{1,J-1,E_1e_2}\cos\omega + pw_{1,J-1,e_2} + qw_{1,J-1}) - \varepsilon^{J+1} w_{1,J-2,e_2e_2}. \tag{3.6}$$

Since $d_1 = (\csc\omega)e_2$ on Γ_1 we have $W_{1,J} = g_{1,1}$ on Γ_1. The function $W_{1,J}$ is non-zero on Γ_2. Letting $g_{2,2}$ denote the trace of $W_{1,J}$ on Γ_2, since $e_2 = 0$ and $e_1 = (\sin\omega)d_2$ on Γ_2 we have

$$g_{2,2}(d_2) = W_{1,J}\big((\sin\omega)d_2/\varepsilon, 0\big).$$

The integer J will be chosen below.

The solution to the two point boundary value problem that defines $w_{1,0}$ is

$$w_{1,0}(E_1, e_2) = g_{1,1}(e_2\csc\omega)e^{-p(\sin\omega_1)E_1}.$$

Since $\omega_1 \in (0,\pi)$, $\sin\omega_1 > 0$ and this formula contains a decaying exponential. Also, we note regarding this solution that

$$\frac{\partial}{\partial d_1} w_{1,0} = (\cos\omega_1)\frac{\partial}{\partial x} w_{1,0} + (\sin\omega_1)\frac{\partial}{\partial y} w_{1,0} = (\sin\omega) w_{1,e_2}.$$

Since w_{1,e_2} is bounded uniformly in ε, it follows that the directional derivative $(\partial/\partial d_1)w_{1,0}$ is bounded uniformly in ε. Similar remarks hold for the other boundary layer functions.

We require bounds on the functions $w_{1,j}$. For this we will use

Lemma 10. *Let $p > 0$, $a > 0$, and let $F(E)$ satisfy $|D_E^\mu F(E)| \le F_m e^{-aE}$ for $\mu = 0, \cdots, m$. Let $W'' + pW' = F$ for $E > 0$ with $W(E) \to 0$ as $E \to \infty$. Then there is a constant $C(a,p)$ such that if $b < \min\{p, a\}$,*

$$|D_E^\mu W(E)| \le C(a,p,\mu)[F_m + |W(0)|]e^{-bE} \text{ for } \mu = 0, \cdots, m+2.$$

If $p \ne a$ one may take $b = \min\{p, a\}$.

Proof. The proof follows from the solution formula

$$W(E) = -p^{-1}e^{-pE}\int_0^E (e^{ps} - 1)F(s)ds - p^{-1}(1 - e^{-pE})\int_E^\infty F(s)ds + W(0)e^{-pE}.$$

In the case $a = p$ the bound involves $(1+E)e^{-pE}$, which we have majorized by Ce^{-bE} for any $b < p$. ■

We now apply Lemma 10 with p and a replaced by $p\sin\omega_1$. An inductive argument gives the exponentially decaying bounds

$$|D_{E_1}^m w_{1,j}(E_1, e_2)| \le C \max_{0\le\mu\le j} |D^\mu g_{1,1}(e_2\csc\omega)|e^{-b(\sin\omega_1)E_1},\ b < p,\ m = 0, 1, \cdots. \tag{3.7}$$

The ordinary differential equations that define the functions $w_{1,j}$ contain e_2 as a parameter. Differentiating these equations with respect to e_2, one obtains the same set of equations, but with different boundary data and right hand sides, for the functions $D_{e_2}^m w_{1,j}(E_1, e_2)$. Applying (3.7) to this system, we obtain

$$|D_{E_1}^{m_1} D_{e_2}^{m_2} w_{1,j}(E_1, e_2)| \le C \max_{0\le\mu\le j+m_2} |D^\mu g_{1,1}(e_2\csc\omega)|e^{-b(\sin\omega_1)E_1},\ b < p,\ m = 0, 1, \cdots. \tag{3.8}$$

In particular, this formula leads to bounds for derivatives of the function $g_{2,2}$. We get from (3.8)

$$|D_{E_1}^m w_{1,j}(E_1, 0)| \le C \max_{0\le\mu\le j} |D^\mu g_{1,1}(0)|e^{-b(\sin\omega_1)E_1},\ b < p,\ m = 0, 1, \cdots.$$

Hence

$$|D^m g_{2,2}(r)| \le C\varepsilon^{-m} e^{-b(\sin\omega)(\sin\omega_1)r/\varepsilon} \quad b < p. \tag{3.9}$$

In a similar manner we will use a function $w_2(e_1, E_2)$ to represent the boundary layer along Γ_2. For derivatives of w_2 we have

$$\frac{\partial}{\partial x} w_2(e_1, E_2) = -w_{2,e_1}\sin\omega_1 + \varepsilon^{-1} w_{2,E_2}\sin\omega_2, \ \frac{\partial}{\partial y} w_2(e_1, E_2) = w_{2,e_1}\cos\omega_1 - \varepsilon^{-1} w_{2,E_2}\cos\omega_2,$$

$$\frac{\partial^2}{\partial x^2} w_2(e_1, E_2) = w_{2,e_1e_1}\sin^2\omega_1 - 2\varepsilon^{-1} w_{2,e_1E_2}\sin\omega_1\sin\omega_2 + \varepsilon^{-2} w_{2,E_2E_2}\sin^2\omega_2,$$

$$\frac{\partial^2}{\partial y^2} w_2(e_1, E_2) = w_{2,e_1e_1}\cos^2\omega_1 - 2\varepsilon^{-1} w_{2,e_1E_2}\cos\omega_1\cos\omega_2 + \varepsilon^{-2} w_{2,E_2E_2}\cos^2\omega_2,$$

$$\begin{aligned} Lw_2 = {} & -\varepsilon^{-1}\left[w_{2,E_2E_2} - p w_{2,E_2}\sin\omega_2\right] \\ & + 2w_{2,e_1E_2}\cos\omega - p w_{2,e_1}\sin\omega_1 + q w_2 - \varepsilon w_{2,e_1e_1}. \end{aligned}$$

We define a sequence of functions $w_{2,j}(e_1, E_2)$ by writing $L(\sum_0 \varepsilon^j w_{2,j}(e_1, E_2)) = 0$ and equating coefficients of ε. In this way we arrive at the sequence of two point boundary value problems

$$w_{2,0,E_2E_2} - p w_{2,0,E_2}\sin\omega_1 = 0, \ w_{2,0}(e_1, 0) = g_{1,2}\big(e_1\csc\omega\big), w_{2,0}(e_1, E_2) \to 0 \text{ as } E_2 \to \infty.$$

$$\begin{aligned} w_{2,1,E_2E_2} - p w_{2,1,E_2}\sin\omega_1 &= 2w_{2,0,e_1E_2}\cos\omega - p w_{2,0,e_1} + q w_{2,0}, \\ w_{2,1}(e_1, 1) &= 0, w_{2,1}(e_1, E_2) \to 0 \text{ as } E_2 \to \infty, \end{aligned}$$

$$\begin{aligned} w_{2,j,E_2E_2} - p w_{2,j,E_2}\sin\omega_1 &= 2w_{2,j-1,e_1E_2}\cos\omega - p w_{2,j-1,e_1} + q w_{2,j-1} - w_{2,j-2,e_1e_1}, \\ w_{2,j}(e_1, 0) &= 0, w_{2,j}(e_1, E_2) \to 0 \text{ as } E_2 \to \infty, \text{ for } j > 1. \end{aligned}$$

Setting $W_{2,J} = \sum_0^J w_{2,j}$ we have

$$LW_{2,J} = \varepsilon^J(2w_{2,J-1,e_1E_2}\cos\omega - p w_{2,J-1,e_1} + q w_{2,J-1}) - \varepsilon^{J+1} w_{2,J-2,e_1e_1}. \tag{3.10}$$

Since $d_2 = (\csc\omega)e_1$ on Γ_2 we have $W_{2,J} = g_{2,2}$ on Γ_2. The function $W_{2,J}$ is non-zero on Γ_1. Letting $g_{1,1}$ denote the trace of $W_{2,J}$ on Γ_1, since $e_1 = 0$ and $e_2 = (\sin\omega)d_1$ on Γ_1 we have

$$g_{1,1}(d_1) = W_{2,J}\big(0, (\sin\omega)d_1/\varepsilon\big).$$

The solution to the two point boundary value problem that defines $w_{2,0}$ is

$$w_{2,0}(e_1, E_2) = g_{1,2}\big(e_1\csc\omega\big)e^{p(\sin\omega_2)E_2}.$$

Since $\omega_2 \in (\pi, 2\pi)$, $\sin\omega_2 < 0$ and this formula contains a decaying exponential. Also, we note regarding this solution that

$$\frac{\partial}{\partial d_2} w_{2,0} = (\cos\omega_2)\frac{\partial}{\partial x} w_{2,0} + (\sin\omega_2)\frac{\partial}{\partial y} w_{2,0} = (\sin\omega) w_{1,e_1}.$$

Since w_{2,e_1} is bounded uniformly in ε, it follows that the directional derivative $(/\partial d_2)\partial w_{2,0}$ is bounded uniformly in ε. Similar remarks hold for the other boundary layer functions.

We now apply Lemma 10 with p and a replaced by $p\sin\omega_2$. An inductive argument gives the exponentially decaying bounds

$$|D^m_{E_2} w_{2,j}(e_1,E_2)| \le C \max_{0\le\mu\le j} |D^\mu_{e_1} g_{1,2}(e_1\csc\omega)| e^{-b|\sin\omega_2|E_2},\ b<p,\ m=0,1,\cdots. \tag{3.11}$$

The ordinary differential equations that define the functions $w_{2,j}$ contain e_1 as a parameter. Differentiating these equations with respect to e_1, one obtains the same set of equations, but with different boundary data and right hand sides, for the functions $D^m_{e_1} w_{2,j}(e_1,E_2)$. Applying (3.11) to this system, we obtain

$$|D^{m_1}_{e_1} D^{m_2}_{E_2} w_{2,j}(e_1,E_2)| \le C \max_{0\le\mu\le j+m_1} |D^\mu_{e_1} g_{1,2}(e_1\csc\omega)| e^{-b|\sin\omega_2|E_2},\ b<p,\ m=0,1,\cdots. \tag{3.12}$$

In particular, this formula leads to bounds for derivatives of the function $g_{1,1}$. We get from (3.11)

$$|D^m_{E_2} w_{2,j}(0,E_2)| \le C \max_{0\le\mu\le j} |D^\mu g_{2,2}(0)| e^{-b(\sin\omega_2)E_2},\ b<p,\ m=0,1,\cdots.$$

Hence

$$|D^m g_{1,1}(r)| \le C\varepsilon^{-m} e^{-b(\sin\omega)(\sin\omega_2)r/\varepsilon},\ b<p. \tag{3.13}$$

Setting $u_2 = u_1 - W_{1,J} - W_{2,J}$ we see that u_2 satisfies the problem

$$\begin{aligned} Lu_2 &= f_2 := f_1 + f_{2,1} + f_{2,2} \text{ in } S,\\ u_2 &= g_{2,l} \text{ on } \Gamma_l,\ l=1,2, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} f_{2,1} &= \varepsilon^J(2w_{1,J-1,E_1e_2}\cos\omega + pw_{1,J-1,e_2} + qw_{1,J-1}) - \varepsilon^{J+1} w_{1,J-2,e_2e_2},\\ f_{2,2} &= \varepsilon^J(2w_{2,J-1,e_1E_2}\cos\omega + pw_{2,J-1,e_1} + qw_{2,J-1}) - \varepsilon^{J+1} w_{2,J-2,e_1e_1}, \end{aligned}$$

The following lemma gives bounds for f_2.

Lemma 11. *One has*

$$\|f_2\|_{s-2} \le C\left[\varepsilon^{K+1}\|f\|_{2K+s} + \varepsilon^{J-s+\frac12}\|f\|_{2K+J+s-\frac12}\right].$$

Proof. Using (3.2) and (3.4),

$$\|f_1\|_{s-2} = \varepsilon^{K+1}\|\Delta v_K\|_{s-2} \le C\varepsilon^{K+1}\|f\|_{2K+s}.$$

For the estimation of $f_{2,l}$ we must estimate the derivatives of order $\le m$ of $w_{l,J-1}$ and $w_{l,J-2}$. To estimate the derivatives of order m we write

$$\begin{aligned} |w_{1,J}|^2_m &\le C \sum_{m_1+m_2=m} \int_0^\infty\int_0^\infty |D^{m_1}_{e_1} D^{m_2}_{e_2} w_{1,J}(E_1,e_2)|^2 de_1de_2\\ &= C \sum_{m_1+m_2=m} \varepsilon^{1-2m_1} \int_0^\infty\int_0^\infty |D^{m_1}_{E_1} D^{m_2}_{e_2} w_{1,J}(E_1,e_2)|^2 dE_1de_2\\ &\le C \sum_{m_1+m_2=m} \varepsilon^{1-2m_1} \int_0^\infty \max_{0\le\mu\le J+m_2} |D^\mu_{e_2} g_{1,1}(e_2\csc\omega)|^2 de_2\\ &\le C \sum_{m_1+m_2=m} \varepsilon^{1-2m_1} \|g_{1,1}\|^2_{J+m_2,R_+}\\ &\le C\varepsilon^{1-2m}\|g_{1,1}\|^2_{J+m,R_+}. \end{aligned}$$

Hence

$$\begin{aligned}\|w_{1,J}\|_m &\le C\varepsilon^{\frac{1}{2}-m}\|g_{1,1}\|_{J+m,R_+}\\ &\le C\varepsilon^{\frac{1}{2}-m}\|g_{1,1}\|_{J+m,R_+}\\ &\le C\varepsilon^{\frac{1}{2}-m}\|V_K\|_{J+m,\Gamma_1}.\end{aligned}$$

But

$$\|V_K\|_{J+m,\Gamma_1} \le C\|V_K\|_{J+m+\frac{1}{2}} \le C\|f\|_{2K+J+m+\frac{1}{2}},$$

so $\|w_{1,J}\|_m \le C\varepsilon^{\frac{1}{2}-m}\|f\|_{2K+J+m+\frac{1}{2}}$. Similar bounds hold for the other terms in $f_{2,l}$, $l = 1, 2$. Combining these inequalities, the lemma is proved. ■

In the problem (3.14) satisfied by u_2, the boundary values $g_{2,l}$ are not small, although they do decay exponentially along the sides Γ_l. We shall deal with this boundary data using a function $Z_J(x, y)$ defined by the following boundary value problem:

$$LZ_J = 0 \text{ in } S,\ Z_J = g_{2,l} \text{ on } \Gamma_l,\ l = 1, 2. \tag{3.15}$$

Setting $u_3 = u_2 - Z_J$ we see that u_3 satisfies the boundary value problem

$$\begin{aligned}Lu_3 &= f_2 \text{ in } S,\\ u_3 &= 0 \text{ on } \Gamma_l,\ l = 1, 2.\end{aligned} \tag{3.16}$$

For this analysis to succeed we shall need bounds for derivatives of the function Z_J. It may seem that the problem (3.15) has the same complexity as our original problem. However the exponential decay in the boundary functions $g_{2,l}$ enables us to obtain suitable derivative bounds. This is done in the next 3 lemmas. The first lemma uses some of the results of Section 2.

Lemma 12. *Let q be a positive bounded function in S with $q_{\min} = \inf_S q > 0$. Let u solve the problem*

$$\begin{aligned}-\Delta u + qu &= f \text{ in } S,\\ u(r\cos\omega_l, r\sin\omega_l) &= g_l(r),\ l = 1, 2.\end{aligned} \tag{3.17}$$

Suppose f and g_l are smooth functions which satisfies for some integer $m \ge 2$

$$|D^j f(x, y)| \le e^{-ar}F_m \text{ for } j = 0, \cdots, m-2,\ |D^j g_l(r)| \le e^{-ar}F_m \text{ for } j = 0, \cdots, m,\ l = 1, 2,$$

and suppose $g_1(0) = g_2(0)$. Let $\alpha = \pi/\omega$. For any b with $b < a$ and $b^2 < q_{\min}$, there is a number $C_m(b)$ such that for $(x, y) \in S$,

$$|D^m u(x, y)| \le \begin{cases} C_m(q_{\min} - b^2)^{-1}(a - b)^{-1}F_m e^{-br}, & m < \alpha,\\ C_m(q_{\min} - b^2)^{-1}(a - b)^{-1}F_m[1 + |\ln r|]e^{-br}, & m = \alpha,\\ C_m(q_{\min} - b^2)^{-1}(a - b)^{-1}F_m[1 + r^{\alpha-m}]e^{-br}, & m > \alpha.\end{cases} \tag{3.18}$$

Proof. The proof is given in a series of steps.

(i) Select a smooth function G in S which agrees with g_l on Γ_l, $l = 1, 2,$. and which has the same exponential decay for large r. Subtracting G from u, it suffices to consider the case of $g_1 = g_2 = 0$.

(ii) Let S_1 denote the truncated sector S consisting of points of S for which $r \le 1$. Applying the corner singularity expansion of Theorem 1 and using Lemma 1, we obtain $u = u_s + u_r$ where u_s is the sum of a finite number of singular functions and where the remainder $u_r \in H^{m+2}(S_1)$ and satisfies $\|u_r\|_{m+2,S_1} \le C_m\|f\|_{m,S_1}$. Each of the coefficients in the singular expansion is also bounded by $C_m\|f\|_{m,S_1}$. The leading singular function is r^α if $\alpha \neq$ integer, and is $r^\alpha \ln r$ if $\alpha =$ integer. Hence in the case $m \neq \alpha$, if $(x, y) \in S_1$ one has using the Sobolev inequality

$$\begin{aligned}
|D^m u(x,y)| &\leq |D^m u_s(x,y)| + |D^m u_r(x,y)| \\
&\leq C_m(1 + r^{\alpha-m})\|f\|_{m,S_1} + C_m\|u_r\|_{m+2,S_1} \\
&\leq C_m(1 + r^{\alpha-m})\|f\|_{m,S_1} + C_m\|f\|_{m,S_1} \\
&\leq C_m(1 + r^{\alpha-m})F_m + C_m F_m.
\end{aligned}$$

This proves (3.18) in the case $m \neq \alpha$ and $(x, y) \in S_1$. The case $m = \alpha$ and $(x, y) \in S_1$ is similar but uses the log singularity. Hence (3.18) is established for $(x, y) \in S_1$.

(iii) From [3, Theorem 1.4.4.3], for $v \in C_0^\infty(S_1)$,

$$\int\int_{S_1} \frac{v^2}{r^2} dxdy \leq C\|v\|_{1,S_1}^2.$$

Hence

$$\int\int_{S_1} \frac{v^2}{r} dxdy \leq C\|v\|_{1,S_1}^2.$$

Since $\int\int_{r>1} r^{-1}v^2 dxdy \leq \|v\|_0^2 \leq \|v\|_{1,S}^2$, we obtain

$$\int\int_S \frac{v^2}{r} dxdy \leq C\|v\|_{1,S}^2 \text{ if } v \in H_0^1(S).$$

(iv) Let b be given as in the statement of the Lemma, and make the change of variables $u = e^{-br}v$. Formally, v satisfies the problem

$$-\Delta v + b(2v_r + \frac{1}{r}v) + (q - b^2)v = \bar{f} \text{ in } S, \; v = 0 \text{ on } \partial S, \tag{3.19}$$

where $\bar{f} = fe^{br}$. We show that the problem (3.19) has a weak solution $v \in H_0^1(S)$. For this consider the bilinear form

$$B(v, w) = \int\int_S [\nabla v \cdot \nabla w + b(2v_r + \frac{1}{r}v)w + (q - b^2)vw] dxdy.$$

Using (iii) and the boundedness of q one obtains the inequality

$$|\int\int_S r^{-1}vw dxdy| \leq \|r^{-1/2}v\|_{0,S}\|r^{-1/2}w\|_{0,S} \leq C\|v\|_{1,S}\|w\|_{1,S}.$$

Hence B is a bounded bilinear form on $H_0^1(S)$. Also, using the formula $2vv_r + r^{-1}v^2 = r^{-1}(rv^2)_r$ and the inequality $q - b^2 \geq q_{\min} - b^2 > 0$, it is seen that for $v \in C_0^\infty(S)$

$$B(v, v) = \int\int_S [|\nabla v|^2 + (q - b^2)v^2] dxdy \geq c(q_{\min} - b^2)\|v\|_{1,S}^2. \tag{3.20}$$

Since $C_0^\infty(S)$ is dense in $H_0^1(S)$, this inequality holds for all $v \in H_0^1(S)$. Hence from the Lax-Milgram lemma, (3.19) has a unique weak solution $v \in H_0^1(S)$. The solution satisfies

$$\|v\|_{1,S} \leq C(q_{\min} - b^2)^{-1}\|\bar{f}\|_{0,S}. \tag{3.21}$$

(Since $a > b$, $\bar{f} \in L_2(S)$.) The function v is smooth in S and satisfies (3.19). Hence the function $e^{-br}v$ solves (3.17), so $e^{-br}v = u$.

(v) From (iv), $v \in H^1(S)$ is a weak solution of the equation $-\Delta v = f_1$ in S with $f_1 = \bar{f} - b(2v_r + r^{-1}v) + (q - b^2)v \in L_2(S)$. Let B be a disk of radius $\frac{1}{2}$ and center at distance > 1 from the origin, and placed so that $B \cap S \neq \emptyset$. Let B' be the disk of radius $2/3$ and the same center as B. From a standard argument using localization one finds that $v \in H^m(B \cap S)$ and

$$\|v\|_{m+2,B\cap S} \leq C_m[\|\bar{f}\|_{m,B'\cap S} + \|v\|_{1,B'\cap S}],\ m = 0, 1, \cdots.$$

The constant C_m does not depend on the location of B. For $\rho > 0$ let S_ρ denote the truncated sector consisting of points in S with $r < \rho$, and let $S_\rho^c = S \setminus S_\rho$. Taking a collection of disks B that cover S_1^c one obtains

$$\|v\|_{m+2,S_1^c} \leq C_m[\|\bar{f}\|_{m,S_{1/3}^c} + \|v\|_{0,S}],\ m = 0, 1, \cdots.$$

Using (3.21) one obtains

$$\|v\|_{m+2,S_1^c} \leq C_m(q_{\min} - b^2)^{-1}[\|\bar{f}\|_{m,S_{1/3}^c} + \|\bar{f}\|_{0,S}],\ m = 0, 1, \cdots. \tag{3.22}$$

If D^γ is a derivative of order $|\gamma| = m$,

$$D^\gamma \bar{f} = \Big(\sum_{|\beta|\leq m} a_{\gamma\beta} D^\beta f\Big) e^{br}, \tag{3.23}$$

where the coefficients $a_{\gamma\beta}$ are sums of polynomials in x and y divided by powers of r. These coefficients are homogeneous functions of degree ≤ 0. Hence using the assumptions on f and the inequality $a > b$, we see that $\|\bar{f}\|_{m-2,S_1^c} \leq C_m(b)F_m$. From Sobolev's inequality we therefore get

$$|D^m v(x,y)| \leq C\|v\|_{m+2,S_1^c} \leq C_m(a-b)^{-1}(q_{\min} - b^2)^{-1}F_m \text{ for } r \geq 1,\ m = 0, 1, \cdots.$$

Writing as in (3.23)

$$D^\gamma u = \Big(\sum_{|\beta|\leq m} b_{\gamma\beta} D^\beta v\Big) e^{-br},$$

we obtain (3.18) for $(x, y) \in S_1^c$. ■

We now apply this result to the convection diffusion problem. We suppose that the right hand side is zero, and that the boundary data are rapidly decaying.

Lemma 13. *Let z be the solution to the problem*

$$\begin{aligned} -\varepsilon\Delta z + pz_x + qz &= 0 \text{ in } S, \\ z(r\cos\omega_l, r\sin\omega_l) &= g_l(r),\ l = 1, 2. \end{aligned} \tag{3.24}$$

Suppose g_1 and g_2 are smooth functions which satisfy for some integer m

$$|D^j g_l(r)| \leq e^{-cr/\varepsilon}F_m, \text{ for } j = 0, \cdots, m+2,\ l = 1, 2, \tag{3.25}$$

and suppose $g_1(0) = g_2(0)$. Then for any b satisfying $0 < b \leq \min\{p, 2c\}$ there is a constant $C_m(b)$ such that

$$|D^m z(x,y)| \leq \begin{cases} C_m(b)\varepsilon^{-m}F_m e^{-(br-px)/(2\varepsilon)-br/2}, & m < \alpha, \\ C_m(b)\varepsilon^{-m}F_m[1 + |\ln(r/(2\varepsilon))|]e^{-(br-px)/(2\varepsilon)-br/2}, & m = \alpha, \\ C_m(b)\varepsilon^{-m}F_m[1 + (r/(2\varepsilon))^{\alpha-m}]e^{-(br-px)/(2\varepsilon)-br/2}, & m > \alpha. \end{cases} \tag{3.26}$$

Proof. Let $d = 1/(2\varepsilon)$, $z = e^{pdx} z_1$. Then $Lz = [-\varepsilon\Delta z_1 + (q + \frac{1}{2}dp^2)z_1]e^{pdx}$ so z_1 satisfies

$$-4\varepsilon^2 \Delta z_1 + (p^2 + 4\varepsilon q)z_1 = 0 \text{ in } S.$$

We now stretch the variables. Let $\xi = dx$, $\eta = dy$, $\rho = (\xi^2 + \eta^2)^{1/2}$, $z_2(\xi, \eta) = z_1(x, y)$. We have

$$\begin{gathered} -\Delta z_2 + (p^2 + 4\varepsilon q)z_2 = 0 \text{ in } S, \\ z_2(\rho\cos\omega_l, \rho\sin\omega_l) = h_l(\rho). \end{gathered} \tag{3.27}$$

The relation between z and z_2 is:

$$z(x, y) = e^{px/(2\varepsilon)} z_1(x, y) = e^{px/(2\varepsilon)} z_2(x/(2\varepsilon), y/(2\varepsilon)) = e^{pdx} z_2(dx, dy). \tag{3.28}$$

Thus the boundary data in (3.27) is given by the formulas

$$\begin{aligned} h_l(\rho) &= z_2(\rho\cos\omega_l, \rho\sin\omega_l) \\ &= z_1(2\varepsilon\rho\cos\omega_l, 2\varepsilon\rho\sin\omega_l) \\ &= e^{-p\xi} z(2\varepsilon\rho\cos\omega_l, 2\varepsilon\rho\sin\omega_l) \\ &= e^{-p\xi} g_l(2\varepsilon\rho) \\ &= e^{-p\rho\cos\omega_l} g_l(2\varepsilon\rho),\ l = 1, 2. \end{aligned}$$

We want to apply Lemma 12 to the problem (3.27). From (3.25), the boundary data of (3.27) satisfies

$$\begin{aligned} |h_l(\rho)| &\le C_0 e^{-(p\cos\omega_l + 2c)\rho} \\ |h_l'(\rho)| &\le p|\cos\omega_l| e^{-p\rho\cos\omega_l} |g_l(2\varepsilon\rho)| + 2\varepsilon e^{-p\rho\cos\omega_l} |g_l'(2\varepsilon\rho)| \\ &\le C_1 e^{-(p\cos\omega_l + 2c)\rho} \end{aligned}$$

and in general,

$$|D^j h_l(\rho)| \le CF_m e^{-(p\cos\omega_l + 2c)\rho},\ j = 0, \cdots, m,\ l = 1, 2.$$

To apply Lemma 12 we must choose b to satisfy $b^2 < p^2 + 4\varepsilon q$ and $b < p\cos\omega_l + 2c$. Since ε may be small, and since we have no control over ω_l, we choose $b \in (0, \min\{p, 2c\})$. We then have

$$|D^m z_2(\xi, \eta)| \le \begin{cases} C_m F_m e^{-b\rho}, & m < \alpha, \\ C_m F_m [1 + |\ln\rho|] e^{-b\rho}, & m = \alpha \text{ integer}, \\ C_m F_m [1 + \rho^{\alpha - m}] e^{-b\rho}, & m > \alpha. \end{cases}$$

Using (3.28) we get (3.26). ∎

To apply Lemma 13 to the function Z_J we note from (3.9) and (3.12) that $c = ps\bar{s}$ where $s = \sin\omega$, $\bar{s} = \min\{\sin\omega_1, \sin\omega_2\}$. Since $x < 0$ in S, (3.26) gives

Lemma 14. *One has*

$$|D^m Z_J(x, y)| \le \begin{cases} C_m \varepsilon^{-m} e^{-ps\bar{s}r/(2\varepsilon) - pr/2}, & m < \alpha, \\ C_m \varepsilon^{-m} [1 + |\ln(r/(2\varepsilon))|] e^{-ps\bar{s}r/(2\varepsilon) - pr/2}, & m = \alpha, \\ C_m \varepsilon^{-m} [1 + (r/(2\varepsilon))^{\alpha - m}] e^{-ps\bar{s}r/(2\varepsilon) - pr/2}, & m > \alpha. \end{cases} \tag{3.29}$$

Using (3.1), (3.7), and (3.10) we have good bounds for Lu_3. We now apply Theorem 2 to the boundary value problem (3.16). Using (2.19) we write

$$u_3 = \sum_{i=1}^{I} \Lambda_{L,i}(f_2)\zeta_i + u_4. \tag{3.30}$$

The number I of terms in the expansion is chosen large enough so that the remainder u_4 has derivatives to the desired order. Combining the various formulas given above we arrive at the following decomposition of the solution u of (1.1):

$$u = V_K + W_{1,J} + W_{2,J} + Z_J + \sum_1^I \Lambda_{L,i}(f_2)\zeta_i + u_4. \tag{3.31}$$

Each term in this decomposition has a role to play in the decomposition. The term V_K coming from the sequence of reduced problems, reduces the size of the right hand side. The terms $W_{1,J}$ and $W_{2,J}$ represent the boundary layers along the two sides. However each of these terms produces a residual exponentially decaying boundary value on the other side. The term Z_J handles these exponentially decaying boundary values. This term produces corner singularities. The sum $\sum \Lambda_{L,i}(f_2)\zeta_i$ subtracts all the corner singularities, enabling the remainder u_4 to be both smooth and small. Derivative bounds are given in the following theorem.

Theorem 3. *There is a constant $a > 0$ such that if $f \in H^s(S)$, for any $s > 0$ there is a constant $C > 0$ such that for $n \le N$ one has the derivative bounds*

$$|D^n u(x,y)| \le C\left[1 + \varepsilon^{-n}e^{-ae_1/\varepsilon} + \varepsilon^{-n}e^{-ae_2/\varepsilon} + (r/\varepsilon)^{\alpha-n}e^{-ar/\varepsilon}\right], \text{ if } \alpha \neq n, \tag{3.32}$$

and for $n = m + k \le N$ one has the more precise derivative bounds

$$\begin{aligned} |D_{d_1}^m D_{e_1}^k u(x,y)| &\le C\left[1 + \varepsilon^{-k}e^{-ae_1/\varepsilon} + \varepsilon^{-n}e^{-ae_2/\varepsilon} + (r/\varepsilon)^{\alpha-n}e^{-ar/\varepsilon}\right], \text{ if } \alpha \neq n, \\ |D_{d_2}^m D_{e_2}^k u(x,y)| &\le C\left[1 + \varepsilon^{-k}e^{-ae_2/\varepsilon} + \varepsilon^{-n}e^{-ae_1/\varepsilon} + (r/\varepsilon)^{\alpha-n}e^{-ar/\varepsilon}\right] \text{ if } \alpha \neq n. \end{aligned} \tag{3.33}$$

In the case $\alpha = n$ the inequalities are modified by replacing $(r/\varepsilon)^{\alpha-n}$ by $|\ln(r/\varepsilon)|$.

Proof. Let N be given. Since we seek bounds for derivatives of u of order $\le N$, and motivated by the Sobolev imbedding theorem, we choose an $s > N + 2$ such that $s_I < s < s_{I+1}$ for some I. Motivated by Lemma 11, and recalling (2.20), we choose J and K so that $K + 1 - N - \sigma_i > 0$, $J - s + \frac{1}{2} - N - \sigma_i > 0$ for $i = 1, \cdots, I$, and $K + 1 - \sigma(s) > 0$, $J - s + \frac{1}{2} - \sigma(s) > 0$. With this choice of I, J, and K, we use the decomposition (3.31). We must show that each of the terms in the decomposition (3.31) satisfies inequalities corresponding to (3.32) and (3.33). Using (3.2) we see that $D^n V_K$ is bounded in S. From (3.8) and (3.12) it is seen that $W_{1,J}$ and $W_{2,J}$ satisfy these inequalities. From (2.16) and (2.17),

$$|D^n \zeta_i(x,y)| \le C\left[1 + \varepsilon^{-n}r^{\alpha-n}e^{-ar/\varepsilon}\right].$$

Hence

$$\begin{aligned} |D^n \Lambda_{L,i}(f_2)\zeta_i| &\le C|\Lambda_{L,i}(f_2)|\left[1 + \varepsilon^{-n}r^{\alpha-n}e^{-ar/\varepsilon}\right] \\ &\le C\varepsilon^{-N-\sigma_i}\|f_2\|_{s-2}\left[1 + r^{\alpha-n}e^{-ar/\varepsilon}\right] \\ &\le C\left(\varepsilon^{K+1-N-\sigma_i} + \varepsilon^{J-s+\frac{1}{2}-N-\sigma_i}\right)\left[1 + r^{\alpha-n}e^{-ar/\varepsilon}\right] \\ &\le C\left[1 + r^{\alpha-n}e^{-ar/\varepsilon}\right]. \end{aligned}$$

Hence the terms in the sum satisfy the inequalities (3.33). Finally, using using (2.20) and the Sobolev inequality, the remainder u_3 is seen to be bounded. ■

.

References

[1] M. Abramowitz and I. Stegun, *Handbook of mathematical functions,*, National Bureau of Standards, 1965.

[2] M. Dauge, *Elliptic boundary value problems on corner domains,* Springer Verlag, Lecture Notes in Mathematics, No. 1341, 1988.

[3] P. Grisvard, *Elliptic problems in nonsmooth domains,* Pitman, 1985.

[4] P. Grisvard, *Singularities in boundary value problems,* Masson, 1992.

[5] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities, Cambridge University Press,*1952.

[6] V. A. Kozlov, V. G. Maz'ya, and J. Rossmann, *Elliptic boundary value problems in domains with point singularities,* American Mathematical Society, 1997.

[7] S. A. Nazarov and B. A. Plamenevskii, *Elliptic problems in domains with piecewise smooth boundaries*, De Gruyter, 1994.