ON THE COMMUTABILITY BETWEEN HYPERSPACES AND SUBSPACES

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Abstract. We study the problem of when the hyperspace $CL(A)$ of a subspace A of a space X is canonically representable as a subspace of the hyperspace $CL(X)$, where both $CL(A)$ and $CL(X)$ are endowed with one of the following hypertopologies: Fell, Wijsman, d-proximal, Hausdorff, locally finite, proximal ball.

1. Introduction

For every topological space X, we denote by $CL(X)$ the collection of all closed non-empty subsets of X. Let $\mathcal{C}$ be a class of topological spaces, closed under subspaces, and let, for every $X \in \mathcal{C}$, $T(X)$ be a fixed topology on $CL(X)$, called *T-hypertopology* on $CL(X)$. For any space $X \in \mathcal{C}$ and any subspace A of X, we denote by $i_{A,X}$ the map between the spaces $(CL(A), T(A))$ and $(CL(X), T(X))$, defined by the formula $i_{A,X}(B) = cl_X B$, for each $B \in CL(A)$. Obviously, for each subspace A of X, the map $i_{A,X}$ is an injection. It is natural to ask when $i_{A,X}$ is a homeomorphic embedding for every subspace A of X, i.e. when the *hyperspace* $CL(A)$ of the *subspace* A of X is canonically representable as a *subspace* of the *hyperspace* $CL(X)$, or, in other words, when the commutability between hyperspaces and subspaces holds. This question was raised implicitly by H.-J.Schmidt in [6] where he proved that, if $\mathcal{C}$ is the

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class of all topological spaces and T is the lower Vietoris hypertopology, then the above map is always a homeomorphic embedding (see [6,Theorem 11(1)]). In its present general form this question was formulated by G.Dimov. He investigated it in [2] for T equal either to the upper Vietoris hypertopology or to the Vietoris hypertopology. We investigate this problem for T equal either to the Fell, or d-proximal, or Wijsman, or proximal ball, or locally finite, or Hausdorff uniform hypertopology answering a question of G. Dimov. As in [2], we study also the following problems: when the map $i_{A,X}$ is a homeomorphic embedding (resp., an (inversely) continuous map) for a fixed subspace A of X?

Now, we will introduce/recall some definitions and notations. We say that an injection $i : A \longrightarrow X$ between two spaces A and X is *inversely (uniformly) continuous*, if the inverse map $j = (i \upharpoonright A)^{-1} : i(A) \longrightarrow A$ is (uniformly) continuous. By $\mathcal{HS}(T,\mathcal{C})/\mathcal{IHS}(T,\mathcal{C})/\mathcal{COM}(T,\mathcal{C})$ we denote the class of all spaces X from the class $\mathcal{C}$ such that for each subspace A of X the map $i_{A,X}$ is continuous/ inversely continuous/a homeomorphic embedding with respect to the hypertopology T. Let $(X,\mathfrak{U})$ be a uniform space, $U \in \mathfrak{U}$, $x \in X$, and $F \subseteq X$. Then

$B_X(x,U) \stackrel{def}{=} \{y \in X : (x,y) \in U\}$

$B_X(F,U) \stackrel{def}{=} \bigcup\limits_{y \in F} B_X(y,U)$.

Let (X,d) be a metric space, $\varepsilon > 0$, $x \in X$, and $F \subseteq X$. Then

$B_X(x,\varepsilon) \stackrel{def}{=} \{y \in X : d(x,y) < \varepsilon\}$

$\overline{B}_X(x,\varepsilon) \stackrel{def}{=} \{y \in X : d(x,y) \leqq \varepsilon\}$

$B_X(F,\varepsilon) \stackrel{def}{=} \bigcup\limits_{y \in F} B_X(y,\varepsilon)$.

For a topological space X and $G \subseteq X$ we set:

$G^- = \{F \in CL(X) : F \cap G \neq \varnothing\}$

$G^+ = \{F \in CL(X) : F \subseteq G\}$.

For a uniform space $(X,\mathfrak{U})$ and $G \subseteq X$ we set:

$G^{++} = \{F \in CL(X) :$ there exists $U \in \mathfrak{U}$ such that $B_X(F,U) \subseteq G\}$.

For a metric space (X,d) and $G \subseteq X$ the last notation transforms into:

$G^{++} = \{F \in CL(X) :$ there exists $\varepsilon > 0$ such that $B_X(F,\varepsilon) \subseteq G\}$.

Let X be a topological space. *The lower Vietoris hypertopology* $\mathcal{V}^-$ on $CL(X)$ has for a subbase the family $\{G^- : G$ is open in $X\}$. *The upper Vietoris hypertopology* $\mathcal{V}^+$ on $CL(X)$ has for a base the family $\{G^+ : G$ is open in $X\}$. *The Vietoris hypertopology* $\mathcal{V}$ on $CL(X)$ is the supremum of $\mathcal{V}^-$ and $\mathcal{V}^+$. The class of all T_i-spaces will be denoted by $\mathcal{T}_i$ (for $i = 0,1,2,3,3\frac{1}{2}$), and the class of all normal spaces – by $\mathcal{N}$ (normal spaces are not assumed to be, in general, T_1-spaces). We denote by $\mathbb{N}$ the set of all natural numbers, and for every $k \in \mathbb{N}$, we set $k\mathbb{N} = \{kn : n \in \mathbb{N}\}$. We denote by $\mathbb{R}$ the real line endowed with its natural topology.

2. A lemma

The following lemma will facilitate our considerations with respect to certain hit-and-miss hypertopologies.

Lemma 2.1. *Let, for every space Z, the T-hypertopology on $CL(Z)$ has the form $T(Z) = T^-(Z) \vee T^+(Z)$, where $T^-(Z)$ is the lower Vietoris hypertopology on $CL(Z)$, and the hypertopology $T^+(Z)$ has for a base the family $BT^+(Z)$ which is closed under finite intersections and satisfies the following conditions:*

($*$) if $\mathbf{V} \in BT^+(Z)$ and $F \in \mathbf{V}$, then $CL(F) \subseteq \mathbf{V}$;

($**$) if $\mathbf{V} \in BT^+(Z)$ and $F_1, F_2 \in \mathbf{V}$, then $F_1 \cup F_2 \in \mathbf{V}$.

Let $X \in \mathcal{T}_1$, and let A be a subspace of X. Then:

(a) the map $i_{A,X} : (CL(A), T(A)) \longrightarrow (CL(X), T(X))$ is continuous iff the map $i_{A,X} : (CL(A), T^+(A)) \longrightarrow (CL(X), T^+(X))$ is continuous;

(b) the map $i_{A,X} : (CL(A), T(A)) \longrightarrow (CL(X), T(X))$ is inversely continuous iff the map $i_{A,X} : (CL(A), T^+(A)) \longrightarrow (CL(X), T^+(X))$ is inversely continuous.

Proof. Let $BT^+(X)$ be a fixed base of the T^+-hypertopology on $CL(X)$ which satisfies the conditions ($*$) and ($**$) and which is closed under finite intersections. Let $BT^-(X)$ be the standard base of the lower Vietoris hypertopology on $CL(X)$, and let $BT(X)$ be the base of the T-hypertopology on $CL(X)$ which consists of all finite intersections of the elements of $BT^+(X) \cup BT^-(X)$. Then the elements of $BT(X)$ are of the form $(\cap_{k=1}^n G_k^-) \cap \mathbf{V}$, where $\mathbf{V}$ is from $BT^+(X)$ and G_k is an open subset of X (for each $k = 1, \ldots, n$). On $CL(A)$, we fix a base $BT^+(A)$ and construct $BT(A)$ analogously.

(a)($\Longrightarrow$) Let the map $i_{A,X}$ be continuous with respect to T. Let $F \in CL(A)$, and let $\mathbf{U} \in BT^+(X)$ be a neighbourhood of $cl_X F$. By the continuity of $i_{A,X}$ with respect to T, there exists a neighbourhood $\mathbf{V}$ of F such that $\mathbf{V} \in BT(A)$ and $i_{A,X}(\mathbf{V}) \subseteq \mathbf{U}$. This neighbourhood $\mathbf{V}$ has the form $\mathbf{V} = (\cap_{k=1}^n G_k^-) \cap \mathbf{W}$, where $\mathbf{W} \in BT^+(A)$ and G_k is an open subset of A (for each $k = 1, \ldots, n$). Obviously, $F \in \mathbf{W}$. We are going to show that $i_{A,X}(\mathbf{W}) \subseteq \mathbf{U}$.

From $F \in \mathbf{V}$, it follows that, for each $k = 1, \ldots, n$, there exists $g_k \in F \cap G_k$. Let $G = \{g_1, \ldots, g_n\}$. Obviously, $G \in CL(A)$. Since $G \in CL(F)$ and $F \in \mathbf{W}$, the condition ($*$) implies that $G \in \mathbf{W}$.

Let now M be an arbitrary element of $\mathbf{W}$. We set $M_1 = M \cup G$. Since M, $G \in \mathbf{W}$, we obtain, by ($**$), that $M_1 \in \mathbf{W}$. In addition, $M_1 \in G_k^-$ for each $k = 1, \ldots, n$. Hence $M_1 \in \mathbf{V}$, and therefore $i_{A,X}(M_1) \in \mathbf{U}$. We have that

$i_{A,X}(M) = cl_X M \subseteq cl_X M_1 = i_{A,X}(M_1)$ and $i_{A,X}(M_1) \in \mathbf{U}$. Now, by $(*)$, we obtain that $i_{A,X}(M) \in \mathbf{U}$. Consequently, $i_{A,X}(\mathbf{W}) \subseteq \mathbf{U}$. Therefore, $i_{A,X}$ is continuous with respect to T^+.□

($\Longleftarrow$) Let the map $i_{A,X}$ be continuous with respect to T^+. Since the map $i_{A,X}$ is continuous with respect to T^- (by the H.-J.Schmidt's theorem [6,Theorem 11(1)]) and $T = T^- \vee T^+$, it is clear that $i_{A,X}$ is continuous with respect to T.□

(b)($\Longrightarrow$) Let the map $i_{A,X}$ be inversely continuous with respect to T. Let $F \in CL(A)$, and let $\mathbf{U} \in BT^+(A)$ be a neighbourhood of F. Since the map $i_{A,X}$ is inversely continuous, there exists a neighbourhood $\mathbf{V}$ of $cl_X F$ such that $\mathbf{V} \in BT(X)$ and $i^{-1}_{A,X}(\mathbf{V}) \subseteq \mathbf{U}$. This neighbourhood $\mathbf{V}$ has the form $\mathbf{V} = (\cap_{k=1}^n G_k^-) \cap \mathbf{W}$, where $\mathbf{W} \in BT^+(X)$ and G_k is an open subset of X (for each $k = 1, \ldots, n$). Obviously, $cl_X F \in \mathbf{W}$. We are going to show that $i^{-1}_{A,X}(\mathbf{W}) \subseteq \mathbf{U}$.

From $cl_X F \in \mathbf{V}$, it follows that $G_k \cap cl_X F \neq \varnothing$. But G_k is open in X, and therefore $G_k \cap F \neq \varnothing$ (for each $k = 1, \ldots, n$). Hence, for each $k = 1, \ldots, n$, there exists $g_k \in F \cap G_k$. Let $G = \{g_1, \ldots, g_n\}$. Obviously, $G \in CL(X)$. Since $G \subseteq cl_X F$ and $cl_X F \in \mathbf{W}$, the condition $(*)$ implies that $G \in \mathbf{W}$.

Let now $cl_X M$ $(M \in CL(A))$ be an arbitrary element of $\mathbf{W} \cap i_{A,X}(CL(A))$. We set $M_1 = G \cup M$. Since $cl_X M_1 = G \cup cl_X M$ and $G, cl_X M \in \mathbf{W}$, we obtain, by $(**)$, that $cl_X M_1 \in \mathbf{W}$. In addition, $cl_X M_1 \in G_k^-$ for each $k = 1, \ldots, n$. Hence $cl_X M_1 \in \mathbf{V}$, and therefore $M_1 = i^{-1}_{A,X}(cl_X M_1) \in \mathbf{U}$. Since $M \in CL(M_1)$ and $M_1 \in \mathbf{U}$, we obtain, by $(*)$, that $M \in \mathbf{U}$. Consequently, $i^{-1}_{A,X}(\mathbf{W}) \subseteq \mathbf{U}$. Therefore, $i_{A,X}$ is inversely continuous with respect to T^+.□

($\Longleftarrow$) Let the map $i_{A,X}$ be inversely continuous with respect to T^+. Since the map $i_{A,X}$ is inversely continuous with respect to T^- (by the H.-J.Schmidt's theorem [6,Theorem 11(1)]) and $T = T^- \vee T^+$, it is clear that $i_{A,X}$ is inversely continuous with respect to T.■

3. Fell hypertopology $\mathcal{F}$

Definition 3.1. Let X be a topological space. *The lower Fell hypertopology* $\mathcal{F}^-$ on $CL(X)$ coincides, by definition, with the lower Vietoris hypertopology $\mathcal{V}^-$ on $CL(X)$. *The upper Fell hypertopology* $\mathcal{F}^+$ on $CL(X)$ has for a base the family $\mathcal{BF}^+(X) = \{G^+ : G \text{ is open in } X \text{ and } X \setminus G \text{ is compact } \}$. *The Fell hypertopology* $\mathcal{F}$ on $CL(X)$ is the supremum of $\mathcal{F}^-$ and $\mathcal{F}^+$.

In this section we will investigate the case $T = \mathcal{F}^+$ (see §1 for T) but, by lemma 2.1, all results formulated here will be true also for the case $T = \mathcal{F}$.

Proposition 3.2. *Let the topological space X is Hausdorff. Then, for every subspace A of X, the map $i_{A,X} : (CL(A), \mathcal{F}^+(A)) \longrightarrow (CL(X), \mathcal{F}^+(X))$ is inversely continuous, i.e. $\mathcal{IHS}(\mathcal{F}^+, \mathcal{T}_2) = \mathcal{T}_2$.*

Proof. Let A be an arbitrary subspace of X. Let $F \in CL(A)$, and let $U^+ \in \mathcal{BF}^+(A)$ be a neighbourhood of F. We set $W = X \setminus (A \setminus U)$. Since $A \setminus U$ is compact and X is Hausdorff, we obtain that W is an open subset of X. In addition, $X \setminus W$ is compact because $X \setminus W = A \setminus U$. Therefore $W^+ \in \mathcal{BF}^+(X)$. An easy verification shows that W^+ is a neighbourhood of $cl_X F$ and that $i^{-1}_{A,X}(W^+) \subseteq U^+$. Therefore, the map $i_{A,X} : (CL(A), \mathcal{F}^+(A)) \longrightarrow (CL(X), \mathcal{F}^+(X))$ is inversely continuous.■

Example 3.3. There exists a topological space $X \in \mathcal{T}_1 \backslash \mathcal{T}_2$ and a subspace A of X such that the map $i_{A,X}$ is not inversely continuous, i.e. $\mathcal{IHS}(\mathcal{F}^+, \mathcal{T}_1) \subsetneqq \mathcal{T}_1$. Let $Y = \mathbb{N}$ and the topology on Y be $\{\varnothing\} \cup \{U \subseteq \mathbb{N} : 2\mathbb{N} \backslash U$ is finite or empty$\}$. Let $X = Y \oplus P$ and $A = 2\mathbb{N} \cup P$, where $P = \{*\}$ is one-point space and $* \notin Y$. Then we assert that $i_{A,X}$ is not inversely continuous.

It is easy to check that Y is not compact and that $2\mathbb{N}$ (as a subspace of Y) is compact. Let's set $F = \{*\}$ and $U = \{*\}$. Then we have that U is an open subset of A and $A \setminus U = 2\mathbb{N}$ is compact. Thus U^+ is an open subset of $(CL(A), \mathcal{F}^+(A))$. Since F is a closed subset of A and $F \subseteq U$, we obtain that U^+ is a neighbourhood of F in $(CL(A), \mathcal{F}^+(A))$.

Let's assume that $i_{A,X}$ is inversely continuous. Then there exists a neighbourhood W^+ of $cl_X F$ such that $W^+ \in \mathcal{BF}^+(X)$ and $i^{-1}_{A,X}(W^+) \subseteq U^+$. Since $X \in \mathcal{T}_1$, we obtain that $W \cap A \subseteq U$, and hence $W \cap 2\mathbb{N} = \varnothing$. The set $W \cap Y$ is open in Y because W is open in X. Consequently, either $W \cap Y = \varnothing$ or $2\mathbb{N} \setminus (W \cap Y)$ is finite. But, since $W \cap 2\mathbb{N} = \varnothing$, we obtain that $2\mathbb{N} \setminus (W \cap Y) = 2\mathbb{N} \setminus W = 2\mathbb{N}$. Therefore $W \cap Y = \varnothing$, i.e. $W \subseteq \{*\}$, and from $cl_X F \subseteq W$, we conclude that $W = \{*\}$. Then $X \setminus W = Y$, and consequently $X \backslash W$ is not compact. This is a contradiction because $W^+ \in \mathcal{BF}^+(X)$. Therefore, $i_{A,X}$ is not inversely continuous.■

Let's remark, that $\mathcal{IHS}(\mathcal{F}^+, \mathcal{T}_1 \setminus \mathcal{T}_2) \neq \varnothing$, i.e. there exists a topological space $X \in \mathcal{T}_1 \setminus \mathcal{T}_2$, such that for every subspace A of X, the map $i_{A,X}$ is inversely continuous (see example 3.10).

Proposition 3.4. *Let X be a Hausdorff space, and let A be a subspace of X. Then $i_{A,X} : (CL(A), \mathcal{F}^+(A)) \longrightarrow (CL(X), \mathcal{F}^+(X))$ is continuous iff A is closed in X.*

Proof. $(\Longrightarrow)$ Let the map $i_{A,X}$ be continuous. Assume that A is not a closed subset of X. Then $A \neq \varnothing$, and we can find a point $y_0 \in X \setminus A$ such that each neighbourhood of y_0 in X to have a non-empty intersection with A. We set $W = X \setminus \{y_0\}$. Then W is open in X and $X \setminus W$ is compact. Therefore W^+ is an open subset of $(CL(X), \mathcal{F}^+(X))$. Let $x_0 \in A$. Then $x_0 \in W$, $\{x_0\} \in CL(A)$, and $cl_X\{x_0\} = \{x_0\} \in W^+$. Since the map $i_{A,X}$ is continuous, we obtain that there exists a neighbourhood $U^+ \in \mathcal{BF}^+(A)$ of $\{x_0\}$ such that $i_{A,X}(U^+) \subseteq W^+$. Since $y_0 \notin A \setminus U$, $A \setminus U$ is compact and $X \in \mathcal{T}_2$, there exist open subsets U_0 and U_1 of X such that $y_0 \in U_0$, $A \setminus U \subseteq U_1$ and $U_0 \cap U_1 = \varnothing$. We set $G = A \setminus U_1$. Then $G \in U^+$, and therefore $cl_X G \in W^+$. We will show that $y_0 \in cl_X G$. Let V be an arbitrary neighbourhood of y_0 in X. We set $V_0 = V \cap U_0$. Then V_0 is a neighbourhood of y_0 in X, and therefore $A \cap V_0 \neq \varnothing$. Since $A = G \cup (A \cap U_1)$ and $(A \cap U_1) \cap V_0 = \varnothing$, we obtain that $A \cap V_0 = G \cap V_0$. Consequently $G \cap V_0 \neq \varnothing$, and hence $G \cap V \neq \varnothing$. Since V was an arbitrary neighbourhood of y_0 in X, we conclude that $y_0 \in cl_X G$, which together with $cl_X G \in W^+$ implies $y_0 \in W$. Thus, we obtain a contradiction, since, by definition, $W = X \setminus \{y_0\}$. Therefore, A is a closed subset of X.□

($\Longleftarrow$) Let A be a closed subset of X. Then it is easy to check that the map $i_{A,X}$ is continuous.■

Corollary 3.5. $\mathcal{HS}(T, \mathcal{T}_2) = \{$*discrete spaces*$\}$ *for* $T \in \{\mathcal{F}^+, \mathcal{F}\}$.

Let's recall the following statement.

Theorem 3.6 (A.H.Stone [7, Theorem 1, (1) and (5)]). *Let X be a topological space. Then X is hereditarily compact iff every strictly decreasing sequence of closed subsets of X is finite.*

Proposition 3.7. *Let $X \in \mathcal{HS}(\mathcal{F}^+, \mathcal{T}_1)$. Then every closed compact subspace of X is hereditarily compact.*

Proof. First we will show that if M is a proper closed compact subspace of X, then M is hereditarily compact. Let $W = X \setminus M$. Then W is a non-empty open subset of X such that $X \setminus W$ is compact. Therefore W^+ is an open subset of $CL(X)$. Let N be an arbitrary subspace of M. We will show that N is compact. We set $A = W \cup N$ and fix a point $x_0 \in W$. Then, since $X \in \mathcal{T}_1$, the set $\{x_0\}$ is closed in A and $cl_X\{x_0\} \in W^+$. The continuity of $i_{A,X}$ implies that there exists a neighbourhood U^+ of $\{x_0\}$ such that $U^+ \in \mathcal{BF}^+(A)$ and $i_{A,X}(U^+) \subseteq W^+$. From the last inclusion, we obtain that $U \subseteq W$ (because $X \in \mathcal{T}_1$). Consequently $N = A \setminus W \subseteq A \setminus U$. On the other hand, we have that $N = M \cap A$, and therefore N is closed in A. Then N is closed in $A \setminus U$, which is compact. Hence, N is compact. Therefore, M is hereditarily compact.

Now we will show that if X is compact, then X is hereditarily compact. By theorem 3.6, it is enough to prove that every strictly decreasing sequence

$M_0 \underset{\neq}{\supseteq} M_1 \underset{\neq}{\supseteq} M_2 \underset{\neq}{\supseteq} \dots$ of closed subsets of X is finite. The space M_1 is a proper closed subset of X and M_1 is compact. Hence, by the first part of our proof, we obtain that M_1 is hereditarily compact. Now, by theorem 3.6, the sequence $M_1 \underset{\neq}{\supseteq} M_2 \underset{\neq}{\supseteq} \dots$ is finite. Therefore, the original sequence is finite, too.■

Example 3.8. We will show that the necessary condition, stated in 3.7, is not sufficient. Let K be the set of the real numbers, endowed with the topology, generated by the base:

$$\{[a, a+\varepsilon) \setminus S : a \in K, \varepsilon > 0 \text{ and } |S| \leqq \aleph_0\}.$$

Since the compact subspaces of K are precisely the finite subspaces, every closed compact subspace of K is hereditarily compact. On the other hand, K is Hausdorff and non-discrete. Therefore, by 3.5, K doesn't belong to the class $\mathcal{HS}(\mathcal{F}^+, \mathcal{T}_2)$. Hence, $K \notin \mathcal{HS}(\mathcal{F}^+, \mathcal{T}_1)$.■

Example 3.9. We will show that $\mathcal{HS}(\mathcal{F}^+, \mathcal{T}_1 \setminus \mathcal{T}_2) \underset{\neq}{\subseteq} \mathcal{T}_1 \setminus \mathcal{T}_2$.

By 3.6, it is enough to construct a space $X \in \mathcal{T}_1 \setminus \mathcal{T}_2$ which has a closed, compact, non-hereditarily compact subspace M. We set $Y = \mathbb{N}$,

$\tau_1 = \{U \subseteq \mathbb{N} : 2\mathbb{N} \setminus U$ is finite or empty$\}$,

$\tau_2 = \{U \subseteq 2\mathbb{N} : 4\mathbb{N} \setminus U$ is finite or empty$\}$,

and $\tau = \tau_1 \cup \tau_2 \cup \{\varnothing\}$.The verification that τ is a topology is strightforward. We set $M = 2\mathbb{N} \cup \{1\}$ and endow M with the topology, induced by τ. Next, we set $X = M \oplus P$, where $P = \{*\}$ is one-point space and $* \notin Y$.Then:

a) M is a closed subset of X;

b) M is compact: let $\{U_\gamma : \gamma \in \Gamma\}$ be a collection of open subsets of Y which covers M. Then $1 \in U_\alpha$ for some $\alpha \in \Gamma$. We have $U_\alpha \neq \varnothing$ and $U_\alpha \nsubseteq 2\mathbb{N}$. Hence $U_\alpha \in \tau_1$, and therefore $2\mathbb{N} \setminus U_\alpha$ is finite or empty. Now, it is easy to find a finite subcovering of M;

c) M is not-hereditarily compact: $\{4\mathbb{N} \setminus \{4k-2\} : k \in \mathbb{N}\}$ is an open covering of $2\mathbb{N}$ which has not a finite subcovering. Therefore, $2\mathbb{N}$ is a non-compact subspace of M.

Finally, it is easy to see that $X \in \mathcal{T}_1 \setminus \mathcal{T}_2$.■

Example 3.10. We will show that $\mathcal{COM}(\mathcal{F}^+, \mathcal{T}_1 \setminus \mathcal{T}_2) \neq \varnothing$. Indeed, every infinite set, endowed with the cofinite topology, belongs to the class $\mathcal{COM}(\mathcal{F}^+, \mathcal{T}_1 \setminus \mathcal{T}_2)$.■

4. D-PROXIMAL HYPERTOPOLOGY δ

Definition 4.1. Let $(X, \mathfrak{U})$ be a uniform space. *The lower d-proximal hypertopology* δ^- on $CL(X)$ coincides, by definition, with the lower Vietoris hypertopology $\mathcal{V}^-$ on $CL(X)$. *The upper d-proximal hypertopology* δ^+ on $CL(X)$ has for a base the family $\{G^{++} : G \text{ is open in } X\}$. *The d-proximal hypertopology* δ on $CL(X)$ is the supremum of δ^- and δ^+.

In this section we will investigate the case $T = \delta^+$ (see §1 for T) but, by lemma 2.1, all results formulated here will be true also for the case $T = \delta$.

Proposition 4.2. *Let* $(X, \mathfrak{U})$ *be a uniform space, and let* $(A, \mathfrak{V})$ *be a subspace of* $(X, \mathfrak{U})$ *(where* $\mathfrak{V}$ *is the restriction of the uniformity* $\mathfrak{U}$ *on* $A \times A$*). Then the map* $i_{A,X} : (CL(A), \delta^+(\mathfrak{V})) \longrightarrow (CL(X), \delta^+(\mathfrak{U}))$ *is:*
(a) continuous;
(b) inversely continuous.

Proof. (a) Let $F \in CL(A)$, and let O^{++} be a neighbourhood of $cl_X F$ in $CL(X)$.
Then there exists $U \in \mathfrak{U}$ such that $B_X(cl_X F, U) \subseteq O$. Choose $W \in \mathfrak{U}$ such that $3W \subseteq U$ and set $Q = A \cap int_X(B_X(cl_X F, W))$. Then Q is open in A, and hence Q^{++} is open in $CL(A)$. We will show that:
1) Q^{++} is a neighbourhood of F in $CL(A)$;
2) $i_{A,X}(Q^{++}) \subseteq O^{++}$.

Proof of 1). We must show only that $F \in Q^{++}$. We choose $U_1 \in \mathfrak{U}$ such that $2U_1 \subseteq W$ and set $V = U_1 \cap (A \times A)$. We will show that $B_A(F, V) \subseteq Q$. Let $h \in B_A(F, V)$. In order to prove that $h \in Q$, it suffices to check that $B_X(h, U_1) \subseteq B_X(cl_X F, W)$. Let $g \in B_X(h, U_1)$. Then $(h, g) \in U_1$. Since $h \in B_A(F, V)$, we can find $f \in F$ such that $(f, h) \in V \subseteq U_1$. From $(h, g) \in U_1$ and $(f, h) \in U_1$, we obtain that $(f, g) \in U_1 + U_1 = 2U_1 \subseteq W$. Therefore, $g \in B_X(f, W)$, and hence $g \in B_X(cl_X F, W)$.□

Proof of 2). Let G be an arbitrary element of Q^{++}. We will show that $B_X(cl_X G, W) \subseteq O$. Let $h \in B_X(cl_X G, W)$. Then there exists $h_1 \in cl_X G$ such that $(h, h_1) \in W$. For this h_1 we can find $g \in G$ such that $g \in B_X(h_1, W)$. Then $(h_1, g) \in W$.

Since $G \in Q^{++}$, we have that $B_A(G, V) \subseteq Q$ for some $V \in \mathfrak{V}$. Then $g \in B_A(G, V) \subseteq B_X(cl_X F, W)$, and therefore we can find $f \in cl_X F$ such that $(g, f) \in W$. Finally, from $(h, h_1) \in W$, $(h_1, g) \in W$ and $(g, f) \in W$, we obtain that $(h, f) \in W + W + W = 3W \subseteq U$. Hence $h \in B_X(f, U) \subseteq B_X(cl_X F, U) \subseteq O$.□

(b) Let $F \in CL(A)$, and let Q^{++} be a neighbourhood of F in $CL(A)$. Then there exists $V \in \mathfrak{V}$ such that $B_A(F,U) \subseteq Q$. Since $\mathfrak{V}$ is the restriction of $\mathfrak{U}$ on $A \times A$, there exists $U \in \mathfrak{U}$ such that $V = U \cap (A \times A)$. Choose $W \in \mathfrak{U}$ such that $3W \subseteq U$ and set $O = int_X(B_X(cl_X F, W))$. Then O is open in X, and hence O^{++} is open in $CL(X)$. We will show that:

1) O^{++} is a neighbourhood of $cl_X F$ in $CL(X)$;

2) $i_{A,X}^{-1}(O^{++}) \subseteq Q^{++}$.

Proof of 1). We must show only that $cl_X F \in O^{++}$. Choose $W_1 \in \mathfrak{U}$ such that $2W_1 \subseteq W$. We will show that $B_X(cl_X F, W_1) \subseteq O$. Let $h \in B_X(cl_X F, W_1)$. For providing that $h \in O$, it suffices to check that $B_X(h, W_1) \subseteq B_X(cl_X F, W)$. Let $g \in B_X(h, W_1)$. Then $(g,h) \in W_1$.Since $h \in B_X(cl_X F, W_1)$ we can find $f \in cl_X F$ such that $(h,f) \in W_1$. From $(g,h) \in W_1$ and $(h,f) \in W_1$, we obtain that $(g,f) \in W_1 + W_1 = 2W_1 \subseteq W$. Therefore, $g \in B_X(f,W) \subseteq B_X(cl_X F, W)$.□

Proof of 2). Let $G \in CL(A)$ be such that $cl_X G \in O^{++}$. We will show that $B_A(G, W \cap (A \times A)) \subseteq Q$. Let $h \in B_A(G, W \cap (A \times A))$. Then there exists $g \in G$ such that $(h,g) \in W$. From $cl_X G \in O^{++}$ and $g \in G$, it follows that there exist $f_1 \in cl_X F$ such that $(g, f_1) \in W$. For this f_1 we can find $f \in F$ such that $f \in B_X(f_1, W)$. Then $(f_1, f) \in W$ .Finally, from $(h,g) \in W$, $(g,f_1) \in W$ and $(f_1, f) \in W$, we obtain that $(h,f) \in W+W+W = 3W \subseteq U$. But $h, f \in A$, and therefore $(h,f) \in U \cap (A \times A) = V$. Hence, $h \in B_A(f,V) \subseteq B_A(F,V) \subseteq Q$.■

Corollary 4.3. $\mathcal{HS}(T, \mathcal{T}_{3\frac{1}{2}}) = \mathcal{IHS}(T, \mathcal{T}_{3\frac{1}{2}}) = \mathcal{COM}(T, \mathcal{T}_{3\frac{1}{2}}) = \mathcal{T}_{3\frac{1}{2}}$ *for* $T \in \{\delta^+, \delta\}$.

5. Hausdorff uniform hypertopology $\mathcal{H}$

Definition 5.1. Let $(X, \mathfrak{U})$ be a uniform space. For every $U \in \mathfrak{U}$ we set $\mathcal{H}(U) = \{(F,G) \in CL(X) \times CL(X) : F \subseteq B_X(G,U) \text{ and } G \subseteq B_X(F,U)\}$. *The Hausdorff uniformity* $\widetilde{\mathfrak{U}}$ on $CL(X)$ has for a uniformity base the collection $\{\mathcal{H}(U) : U \in \mathfrak{U}\}$. *The Hausdorff uniform hypertopology* $\mathcal{H}$ on $CL(X)$ is the topology induced by the Hausdorff uniformity $\widetilde{\mathfrak{U}}$ on $CL(X)$.

In this section the hyperspaces $CL(X)$ and $CL(A)$ will be endowed with the Hausdorff uniformities.

Proposition 5.2. *Let* $(X, \mathfrak{U})$ *be a uniform space, and let* $(A, \mathfrak{V})$ *be a subspace of* $(X, \mathfrak{U})$ *(where* $\mathfrak{V}$ *is the restriction of the uniformity* $\mathfrak{U}$ *on* $A \times A$*). Then the map* $i_{A,X} : (CL(A), \widetilde{\mathfrak{V}}) \longrightarrow (CL(X), \widetilde{\mathfrak{U}})$ *is:*

(a) ([5, Corollary 5.4]) uniformly continuous;

(b) inversely uniformly continuous.

Proof. (b) Let $\mathcal{H}(V)$ be an arbitrary element of the base of the Hausdorff uniformity on $CL(A)$. Then $V \in \mathfrak{V}$ and V has the form $V = W \cap (A \times A)$ for some $W \in \mathfrak{U}$. Let $U \in \mathfrak{U}$ be such that $2U \subseteq W$. We will show that $(i_{A,X} \times i_{A,X})^{-1}(\mathcal{H}(U)) \subseteq \mathcal{H}(V)$.

Let $(cl_X F, cl_X G) \in \mathcal{H}(U)$, where $F, G \in CL(A)$. Let $x \in F$. Then $x \in cl_X F \subseteq B_X(cl_X G, U)$. Hence there exists $z \in cl_X G$ such that $(x, z) \in U$. Since $z \in cl_X G$, we can find $y \in G$ such that $(z, y) \in U$. From $(x, z) \in U$ and $(z, y) \in U$, we obtain that $(x, y) \in U + U = 2U \subseteq W$. Since $x, y \in A$, we have that $(x, y) \in W \cap (A \times A) = V$. Consequently, $x \in B_A(y, V) \subseteq B_A(G, V)$. Thus, we showed that $F \subseteq B_A(G, V)$. In a similar way we prove that $G \subseteq B_A(F, V)$. Therefore, $(F, G) \in \mathcal{H}(V)$.■

Corollary 5.3. $\mathcal{HS}(\mathcal{H}, \mathcal{T}_{3\frac{1}{2}}) = \mathcal{IHS}(\mathcal{H}, \mathcal{T}_{3\frac{1}{2}}) = \mathcal{COM}(\mathcal{H}, \mathcal{T}_{3\frac{1}{2}}) = \mathcal{T}_{3\frac{1}{2}}$

6. Locally finite hypertopology $\mathcal{LF}$

Definition 6.1. Let X be a topological space. *The lower locally finite hypertopology* $\mathcal{LF}^-$ on $CL(X)$ has for a base all sets of the form $\mathcal{O}^- = \{F \in CL(X) : F \cap G \neq \varnothing$ for every $G \in \mathcal{O}\}$, where $\mathcal{O}$ is an open locally finite family. *The upper locally finite hypertopology* $\mathcal{LF}^+$ on $CL(X)$ coicides, by definition, with the upper Vietoris hypertopology $\mathcal{V}^+$ on $CL(X)$. *The locally finite hypertopology* $\mathcal{LF}$ on $CL(X)$ is the supremum of $\mathcal{LF}^-$ and $\mathcal{LF}^+$.

Proposition 6.2. *Let X be a topological space, and let A be a subspace of X. Then the map $i_{A,X} : (CL(A), \mathcal{LF}^-(A)) \longrightarrow (CL(X), \mathcal{LF}^-(X))$ is cotinuous.*

Proof. Let $F \in CL(A)$. Let us fix a neighbourhood of $cl_X F$ from the base of $\mathcal{LF}^-$-hypertopology on $CL(X)$. This neighbourhood has the form $\mathcal{O}^-$ where $\mathcal{O} = \{O_\gamma : \gamma \in \Gamma\}$ is an open locally finite family in X. Then $\mathcal{Q} = \{O_\gamma \cap A : \gamma \in \Gamma\}$ is an open locally finite family in A. It is easy to check that $F \in \mathcal{Q}^-$ and $i_{A,X}(\mathcal{Q}^-) \subseteq \mathcal{O}^-$. Therefore, the map $i_{A,X}$ is continuous with respect to $\mathcal{LF}^-$.■

Proposition 6.3. *Let $X \in \mathcal{T}_1$, and let A be a subspace of X. Then:*

(a) the map $i_{A,X} : (CL(A), \mathcal{LF}(A)) \longrightarrow (CL(X), \mathcal{LF}(X))$ is continuous iff the map $i_{A,X} : (CL(A), \mathcal{LF}^+(A)) \longrightarrow (CL(X), \mathcal{LF}^+(X))$ is continuous;

(b) the map $i_{A,X} : (CL(A), \mathcal{LF}(A)) \longrightarrow (CL(X), \mathcal{LF}(X))$ is inversely continuous iff both maps $i_{A,X} : (CL(A), \mathcal{LF}^+(A)) \longrightarrow (CL(X), \mathcal{LF}^+(X))$ and $i_{A,X} : (CL(A), \mathcal{LF}^-(A)) \longrightarrow (CL(X), \mathcal{LF}^-(X))$ are inversely continuous.

Proof. (a)($\Longrightarrow$) Let the map $i_{A,X}$ be continuous with respect to $\mathcal{LF}$. Let $F \in CL(A)$, and let U^+ be a neighbourhood of $cl_X F$ from the base of $\mathcal{LF}^+$-hypertopology on $CL(X)$. By the continuity of $i_{A,X}$ with respect to $\mathcal{LF}$, there exist an open subset V of A and an open locally finite family $\mathcal{O}$ in A such that $F \in \mathcal{O}^- \cap V^+$ and $i_{A,X}(\mathcal{O}^- \cap V^+) \subseteq U^+$. Obviously $F \in V^+$. We will show that $i_{A,X}(V^+) \subseteq U^+$.

Let $\mathcal{O} = \{O_\gamma : \gamma \in \Gamma\}$. For every $\gamma \in \Gamma$ we choose a point $x_\gamma \in F \cap O_\gamma$ and set $C = \{x_\gamma : \gamma \in \Gamma\}$. Then $C \in CL(A)$. Indeed, let $x \in A \setminus C$. Since $\mathcal{O}$ is a locally finite family, there exists a neighbourhood W_x of x in A such that the set $\Gamma_0 = \{\gamma \in \Gamma : W_x \cap O_\gamma \neq \varnothing\}$ is finite. Then, since $A \in \mathcal{T}_1$, the set $W \overset{def}{=} (\cap_{\gamma \in \Gamma_0} A \setminus \{x_\gamma\}) \cap W_x$ is open in A, and $x \in W$, but $W \cap C = \varnothing$.

Let now G be an arbitrary element of V^+. We set $G_1 = G \cup C$. Then $G_1 \in V^+$. In addition, for every $\gamma \in \Gamma$, $G_1 \cap O_\gamma \supseteq \{x_\gamma\} \neq \varnothing$. Consequently $G_1 \in \mathcal{O}^- \cap V^+$ and therefore $cl_X G_1 \in U^+$. Hence $cl_X G \subseteq cl_X G_1 \subseteq U$. Therefore, $i_{A,X}(G) = cl_X G \in U^+$.$\square$

($\Longleftarrow$) Let the map $i_{A,X}$ be continuous with respect to $\mathcal{LF}^+$. By 6.2 the map $i_{A,X}$ is continuous with respect to $\mathcal{LF}^-$. Therefore, since $\mathcal{LF} = \mathcal{LF}^- \vee \mathcal{LF}^+$, it is clear that $i_{A,X}$ is continuous with respect to $\mathcal{LF}$.$\square$

(b)($\Longrightarrow$) Let the map $i_{A,X}$ be inversely continuous with respect to $\mathcal{LF}$. Let $F \in CL(A)$, and let V^+ be a neighbourhood of F from the base of $\mathcal{LF}^+$-hypertopology on $CL(A)$. Since the map $i_{A,X}$ is inversely continuous with respect to $\mathcal{LF}$, there exist an open subset U of X and an open locally finite family $\mathcal{O} = \{O_\gamma : \gamma \in \Gamma\}$ in X such that $cl_X F \in \mathcal{O}^- \cap U^+$ and $i^{-1}_{A,X}(\mathcal{O}^- \cap U^+) \subseteq V^+$. Obviously $cl_X F \in U^+$. We will show that $i^{-1}_{A,X}(U^+) \subseteq V^+$.

From $cl_X F \in \mathcal{O}^-$ it follows that $cl_X F \cap O_\gamma \neq \varnothing$. But O_γ is open in X and therefore $F \cap O_\gamma \neq \varnothing$. For every $\gamma \in \Gamma$ we choose a point $x_\gamma \in F \cap O_\gamma$ and set $C = \{x_\gamma : \gamma \in \Gamma\}$.

Let now $cl_X G$ ($G \in CL(A)$) be an arbitrary element of U^+. We set $G_1 = G \cup C$. Then $cl_X G_1 = cl_X G \cup cl_X C \subseteq cl_X G \cup cl_X F \subseteq U$, and therefore $cl_X G_1 \in U^+$. In addition, for every $\gamma \in \Gamma$, $cl_X G_1 \cap O_\gamma \supseteq \{x_\gamma\} \neq \varnothing$. Consequently $cl_X G_1 \in \mathcal{O}^-$. From $cl_X G_1 \in \mathcal{O}^- \cap U^+$ it follows that $G_1 \in V^+$. Hence $G \subseteq G_1 \subseteq V$. Therefore $i^{-1}_{A,X}(cl_X G) = G \in V^+$.

Thus we proved that the map $i_{A,X}$ is inversely continuous with respect to $\mathcal{LF}^+$. Now we will show that the map $i_{A,X}$ is inversely continuous with respect to $\mathcal{LF}^-$.

Let $F \in CL(A)$, and let us fix a neighbourhood of F from the base of $\mathcal{LF}^-$-hypertopology on $CL(A)$. This neighbourhood has the form $\mathcal{O}^-$ where $\mathcal{O} = \{O_\gamma : \gamma \in \Gamma\}$ is an open locally finite family in A. Since the map $i_{A,X}$ is inversely continuous with respect to $\mathcal{LF}$, there exist an open subset U of X and an open locally finite family $\mathcal{Q} = \{Q_\lambda : \lambda \in \Lambda\}$ in X such that $cl_X F \in \mathcal{Q}^- \cap U^+$ and $i_{A,X}^{-1}(\mathcal{Q}^- \cap U^+) \subseteq \mathcal{O}^-$. We set $\mathcal{P} = \{Q_\lambda \cap U : \lambda \in \Lambda\}$. Then $\mathcal{P}$ is an open locally finite family in X and $cl_X F \in \mathcal{P}^-$. We will show that $i_{A,X}^{-1}(\mathcal{P}^-) \subseteq \mathcal{O}^-$.

First we will prove the following statement: $(*)$ for every $\gamma \in \Gamma$ there exists $\lambda \in \Lambda$ such that $Q_\lambda \cap U \cap A \subseteq O_\gamma$. Suppose that there exists $\gamma \in \Gamma$ such that for every $\lambda \in \Lambda$ the set $(Q_\lambda \cap U \cap A) \setminus O_\gamma$ is non-empty. Then, for every $\lambda \in \Lambda$, we choose a point $x_\lambda \in (Q_\lambda \cap U \cap A) \setminus O_\gamma$ and set $C = \{x_\lambda : \lambda \in \Lambda\}$. Since $X \in \mathcal{T}_1$ and $\mathcal{Q}$ is a locally finite family in X, we easily obtain that $C \in CL(X)$. Hence $C \in CL(A)$. Obviously $cl_X C = C \in \mathcal{Q}^- \cap U^+$, but $i_{A,X}^{-1}(cl_X C) = C \notin \mathcal{O}^-$ because $C \cap O_\gamma = \varnothing$. This is a contradiction because $i_{A,X}^{-1}(\mathcal{Q}^- \cap U^+) \subseteq \mathcal{O}^-$. Thus, statement $(*)$ is proved.

Let now $cl_X G$ $(G \in CL(A))$ be an arbitrary element of $\mathcal{P}^-$. Let $\gamma \in \Gamma$. By statement $(*)$, there exists $\lambda \in \Lambda$ such that $Q_\lambda \cap U \cap A \subseteq O_\gamma$. From $cl_X G \in \mathcal{P}^-$ it follows that $cl_X G \cap (Q_\lambda \cap U) \neq \varnothing$. But $Q_\lambda \cap U$ is open in X, and therefore $G \cap (Q_\lambda \cap U) \neq \varnothing$. Since $G \subseteq A$, we obtain that $\varnothing \neq G \cap (Q_\lambda \cap U) = G \cap (Q_\lambda \cap U \cap A) \subseteq G \cap O_\gamma$. Consequently, for every $\gamma \in \Gamma$, $G \cap O_\gamma \neq \varnothing$ i.e. $G \in \mathcal{O}^-$. Therefore, $i_{A,X}^{-1}(\mathcal{P}^-) \subseteq \mathcal{O}^-$.□

$(\Longleftarrow)$ Let the map $i_{A,X}$ be inversely continuous with respect to both $\mathcal{LF}^-$ and $\mathcal{LF}^+$. Since $\mathcal{LF} = \mathcal{LF}^- \vee \mathcal{LF}^+$, it is clear that the map $i_{A,X}$ is inversely continuous with respect to $\mathcal{LF}$.■

Corollary 6.4. *(a)* $\mathcal{HS}(\mathcal{LF}, \mathcal{T}_1) = \mathcal{HS}(\mathcal{V}, \mathcal{T}_1) \supseteq \mathcal{T}_4$;
(b) $\mathcal{IHS}(\mathcal{LF}, \mathcal{T}_1) = \mathcal{IHS}(\mathcal{LF}^-, \mathcal{T}_1) \cap \mathcal{IHS}(\mathcal{V}, \mathcal{T}_1)$;
(c) $\mathcal{COM}(\mathcal{LF}, \mathcal{T}_1) = \mathcal{IHS}(\mathcal{LF}^-, \mathcal{T}_1) \cap \mathcal{COM}(\mathcal{V}, \mathcal{T}_1)$.

Proof. By 6.3 and 2.1, we immediately obtain all equalities, and by $\mathcal{N} \subseteq \mathcal{HS}(\mathcal{V}, \mathcal{T}_0)$ [1, 1.3], we obtain the inclusion $\mathcal{T}_4 \subseteq \mathcal{HS}(\mathcal{V}, \mathcal{T}_1)$.■

The class $\mathcal{HS}(\mathcal{V}, \mathcal{T}_2)$ was extensively investigated in [1] and [2]. In [1] was given an internal characterization of this class. In addition, [2] provided characterizations of the classes $\mathcal{IHS}(\mathcal{V}, \mathcal{T}_1)$ and $\mathcal{COM}(\mathcal{V}, \mathcal{T}_4)$.

Example 6.5. We will show that $\mathcal{COM}(\mathcal{LF}, \mathcal{T}_2) \subsetneqq \mathcal{COM}(\mathcal{V}, \mathcal{T}_2)$. By 6.4(c), it is clear that $\mathcal{COM}(\mathcal{LF}, \mathcal{T}_2) \subseteq \mathcal{COM}(\mathcal{V}, \mathcal{T}_2)$. We will describe a space $\Sigma \in \mathcal{COM}(\mathcal{V}, \mathcal{T}_2) \setminus \mathcal{COM}(\mathcal{LF}, \mathcal{T}_2)$.

Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$, let $\Sigma = \mathbb{N} \cup \{\sigma\}$ (where $\sigma \notin \mathbb{N}$) and define a topology on Σ as follows: all points of $\mathbb{N}$ are isolated, and the neighbourhoods of σ are the sets $U \cup \{\sigma\}$ for $U \in \mathcal{F}$. By [2, 2.14], $\Sigma \in \mathcal{COM}(\mathcal{V}, \mathcal{T}_2)$. On the other hand $\Sigma \notin \mathcal{COM}(\mathcal{LF}, \mathcal{T}_2)$. Indeed, let us set $A = \mathbb{N}$. Then $\mathcal{O} = \{\{x\} : x \in A\}$

is an open locally finite family in A and $A \in \mathcal{O}^-$. Let us assume that the map $i_{A,X}$ is inversely continuous with respect to $\mathcal{LF}^-$. Then there exists an open locally finite family $\mathcal{Q} = \{Q_\lambda : \lambda \in \Lambda\}$ in X such that $X = cl_X A \in \mathcal{Q}^-$ and $i_{A,X}^{-1}(\mathcal{Q}^-) \subseteq \mathcal{O}^- = A$. By statement $(*)$ from the proof of 6.3(b)($\Longrightarrow$) it follows that for every $x \in A$ there exists $\lambda(x) \in \Lambda$ such that $Q_{\lambda(x)} \cap A \subseteq \{x\}$ i.e. such that $Q_{\lambda(x)} = \{x\}$ or $Q_{\lambda(x)} = \{x, \sigma\}$. Now it is clear that every neighbourhood of the point σ intersects infinite number of elements of $\mathcal{Q}$. Consequently, the family $\mathcal{Q}$ is not locally finite. This is a contradiction which shows that the map $i_{A,X}$ is not inversely continuous with respect to $\mathcal{LF}^-$. Therefore $\Sigma \notin \mathcal{COM}(\mathcal{LF}^-, \mathcal{T}_2)$. Hence, by 6.3(b), it follows that $\Sigma \notin \mathcal{COM}(\mathcal{LF}, \mathcal{T}_2)$.■

Example 6.6. The class $\mathcal{COM}(\mathcal{LF}, \mathcal{T}_1)$ is non-empty. Indeed, it is easy to check that every discrete space as well as every infinite set, endowed with the cofinite topology, belong to the class $\mathcal{COM}(\mathcal{LF}, \mathcal{T}_1)$.■

7. Wijsman hypertopology $\mathcal{W}$

Definition 7.1. Let $(X, \mathfrak{U})$ be a uniform space, and let the uniformity $\mathfrak{U}$ be generated by a family $\mathfrak{P}$ of pseudometrics on X. *The Wijsman hypertopology* $\mathcal{W}$ on $CL(X)$ is the weakest topology on $CL(X)$, such that for each $x \in X$ and for each pseudometric $\rho \in \mathfrak{P}$, the map $F \mapsto \rho(x, F)$, from $CL(X)$ to $\mathbb{R}$, is continuous. For a metric space (X, d) the above definition transforms into the following one: *The lower Wijsman hypertopology* $\mathcal{W}^-$ on $CL(X)$ coincides, by definition, with the lower Vietoris hypertopology $\mathcal{V}^-$ on $CL(X)$; *the upper Wijsman hypertopology* $\mathcal{W}^+$ on $CL(X)$ has for a subbase the family $\{\mathcal{W}_X(x, r) : x \in X, r \in \mathbb{R}^+\}$, where $\mathcal{W}_X(x, r) = \{F \in CL(X) :$ there exist a real number $R = R(F)$ such that $R > r$ and $F \cap B_X(x, R) = \varnothing\}$; *the Wijsman hypertopology* $\mathcal{W}$ on $CL(X)$ is the supremum of $\mathcal{W}^-$ and $\mathcal{W}^+$.

In this section (except in 7.2) we will investigate the case $T = \mathcal{W}^+$ (see §1 for T) but, by lemma 2.1, all results formulated here will be true also for the case $T = \mathcal{W}$.

Proposition 7.2. *Let $(X, \mathfrak{U})$ be a uniform space, and let the uniformity $\mathfrak{U}$ be generated by a family $\mathfrak{P}$ of pseudometrics. Let $(A, \mathfrak{V})$ be a subspace of $(X, \mathfrak{U})$ (where the uniformity $\mathfrak{V}$ is generated by the family $\mathfrak{P}_A = \{\rho | A \times A : \rho \in \mathfrak{P}\}$). Then the map $i_{A,X} : (CL(A), \mathcal{W}(\mathfrak{V})) \longrightarrow (CL(X), \mathcal{W}(\mathfrak{U}))$ is inversely continuous.*

Proof. For every $x \in X$ (resp. $x \in A$) and for every $\rho \in \mathfrak{P}$ we denote by $g_{x,\rho}$ (resp. $f_{x,\rho}$) the map from $i_{A,X}(CL(A))$ (resp. $CL(A)$) to $\mathbb{R}$, defined by the formula $g_{x,\rho}(F) = \rho(x, F)$ (resp. $f_{x,\rho}(F) = \rho(x, F)$).

Then, by definition of the Wijsman hypertopology, it follows that the maps $g_{x,\rho}$ ($x \in X$, $\rho \in \mathfrak{P}$) and the maps $f_{x,\rho}$ ($x \in A$, $\rho \in \mathfrak{P}$) are continuous. In addition, for fixed $x \in A$ and $\rho \in \mathfrak{P}$ the diagram:

$$\begin{array}{ccc} (i_{A,X}(CL(A)),\mathcal{W}(\mathfrak{U})) & \xrightarrow{i_{A,X}^{-1}} & (CL(A),\mathcal{W}(\mathfrak{V})) \\ g_{x,\rho}\Big\downarrow & & \Big\downarrow f_{x,\rho} \\ \mathbb{R} & =\!= & \mathbb{R} \end{array}$$

is commutative. Really, for an arbitrary $F \in CL(A)$, we have $g_{x,\rho}(cl_X F) = \rho(x, cl_X F) = \rho(x, F) = f_{x,\rho}(F) = f_{x,\rho}(i_{A,X}^{-1}(cl_X F))$.

Let $S = \{f_{x,\rho}^{-1}(V) : x \in A, \rho \in \mathfrak{P}$ and V is open in $\mathbb{R}\}$. Then S is a subbase of the Wijsman hypertopology on $CL(A)$. We will show that $i_{A,X}(U)$ is open in $i_{A,X}(CL(A))$ for each $U \in S$.

Let $U \in S$. Then $U = f_{x,\rho}^{-1}(V)$ for some open subset V of $\mathbb{R}$, $x \in A$, and $\rho \in \mathfrak{P}$. Consequently $i_{A,X}(U) = i_{A,X}(f_{x,\rho}^{-1}(V)) = g_{x,\rho}^{-1}(V)$. Since $g_{x,\rho}$ is continuous, the set $g_{x,\rho}^{-1}(V)$ is open in $i_{A,X}(CL(A))$. Hence the set $i_{A,X}(U)$ is open in $i_{A,X}(CL(A))$. Therefore, $i_{A,X}$ is inversely continuous.■

Lemma 7.3. *Let (X, d) be a metric space, A be a subset of X, and M be an open totally bounded subspace of X such that $M \cap cl_X A \neq \varnothing$. Then, for every $\varepsilon > 0$, there exists a finite subset C of $M \cap A$, which is ε-dense in the space $M \cap cl_X A$.*

Proof. Let G be a finite, $\varepsilon/2$-dense subset of $M \cap cl_X A$. For each $g \in G$ we have $M \cap B_X(g, \varepsilon/2) \cap cl_X A \supseteq \{g\} \neq \varnothing$. Since $M \cap B_X(g, \varepsilon/2)$ is open in X, we obtain that $M \cap B_X(g, \varepsilon/2) \cap A \neq \varnothing$. Then, for each $g \in G$, we choose a point $c(g) \in M \cap B_X(g, \varepsilon/2) \cap A$ and put $C = \{c(g) : g \in G\}$. This C is a finite subset of $M \cap A$ and it is easy to check that C is ε-dense in $M \cap cl_X A$.■

Let's recall the following

Definition. Let (X, d) be a metric space. The metric d is called *B-TB (ball-totally bounded)* if every open d-ball, except X eventually, is totally bounded.

Obviously, every totally bounded metric is B-TB. The converse is not true. For example, the standard Euclidean metric on $\mathbb{R}^n$ ($n \in \mathbb{N}$) is B-TB, but not totally bounded.

Definition. Let (X, d) be a metric space. The metric d is called an *ultrametric* if it is satisfies the following stronger form of the triangle inequality:

for every $x, y, z \in X$, $d(x, y) \leqq max\{d(x, y), d(y, z)\}$.

Proposition 7.4. *Let (X, d) be a metric space. If the metric d is either B-TB or ultrametric, then, for every subspace A of X, the map* $i_{A,X} : (CL(A), \mathcal{W}^+(d|A \times A)) \longrightarrow (CL(X), \mathcal{W}^+(d))$ *is continuous.*

Proof. Let $F \in CL(A)$, and let $\mathbf{U}$ be a neighbourhood of $cl_X F$ from the base of $\mathcal{W}^+$-hypertopology on $CL(X)$. We are going to find a neighbourhood $\mathbf{V}$ of F in $CL(A)$ such that $i_{A,X}(\mathbf{V}) \subseteq \mathbf{U}$.

The neighbourhood $\mathbf{U}$ has the form $\cap_{k=1}^m \mathcal{W}_X(x_k, r_k)$ for some $x_k \in X$ and $r_k \geqq 0$, $(k = 1, \ldots, m)$. Let k be fixed. Since $cl_X F \in \mathcal{W}_X(x_k, r_k)$, we can find a real number $R_k > r_k$ such that $cl_X F \cap B_X(x_k, R_k) = \varnothing$. We choose a real number l_k such that $R_k > l_k > r_k$.

If $cl_X A \cap B_X(x_k, l_k) = \varnothing$, then $i_{A,X}(CL(A)) \subseteq \mathcal{W}_X(x_k, r_k)$, and therefore we can ignore the ingredient $\mathcal{W}_X(x_k, r_k)$ of $\mathbf{U}$. Consequently, there is no loss of generality in assuming that $cl_X A \cap B_X(x_k, l_k) \neq \varnothing$.

Let's regard first the case when d is B-TB. Since F is non-empty and $cl_X F \cap B_X(x_k, l_k) = \varnothing$, we obtain that $B_X(x_k, l_k) \neq X$. Therefore, $B_X(x_k, l_k)$ is totally bounded. We set $\varepsilon_k = R_k - l_k > 0$. Then, by lemma 7.3 (we must set there $X = X, A = A, M = B_X(x_k, l_k), \varepsilon = \varepsilon_k$ and $C = C_k$), there exists a finite subset C_k of $A \cap B_X(x_k, l_k)$, which is ε_k-dense in $cl_X A \cap B_X(x_k, l_k)$. We set $\mathbf{V}_k = \cap\{\mathcal{W}_A(c, \varepsilon_k) : c \in C_k\}$ and $\mathbf{V} = \cap_{k=1}^m \mathbf{V}_k$. Then $\mathbf{V}$ is open in $CL(A)$.

For fixed k and c ($c \in C_k$) we set $\varepsilon = R_k - d(c, x_k)$. Then $\varepsilon > \varepsilon_k$ and $F \cap B_A(c, \varepsilon) = \varnothing$. Hence $F \in \mathcal{W}_A(c, \varepsilon_k)$. Therefore, $F \in \mathbf{V}$.

Let $G \in \mathbf{V}$. Then, for fixed k and c ($c \in C_k$), we have that $G \in \mathcal{W}_A(c, \varepsilon_k)$. Hence $G \cap B_A(c, \varepsilon_k) = \varnothing$ and, since $G \subseteq A$, we obtain that $G \cap B_X(c, \varepsilon_k) = \varnothing$. Then $cl_X G \cap B_X(c, \varepsilon_k) = \varnothing$ too, because $B_X(c, \varepsilon_k)$ is open in X. Consequently $cl_X G \cap (\cup_{c \in C_k} B_X(c, \varepsilon_k)) = \varnothing$. But $cl_X A \cap B_X(x_k, l_k) \subseteq \cup_{c \in C_k} B_X(c, \varepsilon_k)$, and therefore $cl_X G \cap cl_X A \cap B_X(x_k, l_k) = \varnothing$. Hence $cl_X G \cap B_X(x_k, l_k) = \varnothing$ and, since $l_k > r_k$, we obtain that $cl_X G \in \mathcal{W}_X(x_k, r_k)$. Hence, $cl_X G \in \mathbf{U}$.

Let's now regard the case when d is an ultrametric. Then, for every $k = 1, \ldots, m$, we choose a point $a_k \in A \cap B_X(x_k, l_k)$ and set $\mathbf{V} = \cap_{k=1}^m \mathcal{W}_A(a_k, l_k)$. Since $a_k \in B_X(x_k, R_k)$ and d is an ultrametric, we have $B_X(a_k, R_k) = B_X(x_k, R_k)$. Therefore $F \cap B_A(a_k, R_k) \subseteq cl_X F \cap B_X(x_k, R_k) = \varnothing$ and, since $R_k > l_k$, we obtain that $F \in \mathcal{W}_A(a_k, l_k)$ (for each $k = 1, \ldots, m$). Therefore, $F \in \mathbf{V}$.

Let $G \in \mathbf{V}$. Then $G \cap B_X(a_k, l_k) = \varnothing$. Consequently $\varnothing = cl_X G \cap B_X(a_k, l_k) = cl_X G \cap B_X(x_k, l_k)$ and, since $l_k > r_k$, we conclude that $cl_X G \in \mathcal{W}_X(x_k, r_k)$. Thus $cl_X G \in \mathbf{U}$. Therefore, $i_{A,X}(\mathbf{V}) \subseteq \mathbf{U}$.■

Proposition 7.5. *Let $(X, \| \ \|)$ be a linear normed space (over $\mathbb{R}$ or $\mathbb{C}$), d be the metric, induced by the norm, and, for every subspace A of X, the map*

$i_{A,X} : (CL(A), \mathcal{W}^+(d|A \times A)) \longrightarrow (CL(X), \mathcal{W}^+(d))$ *be continuous. Then the metric* d *is B-TB.*

Proof. We need a lemma first.

Lemma 7.6. *In the assumptions of proposition 7.5, let* $\varepsilon > 0$*, and let* $D = \{y \in X : r \leqq d(x,y) < R\}$*, where* $x \in X$*,* r *and* R *are possitive real numbers such that* $0 < R - r < \varepsilon/4$*. Then there exists a finite* ε*-dense subset of* D*.*

Proof of Lemma 7.6. We set $A = X \setminus B_X(x,r)$ and $F = X \setminus B_X(x, R+\varepsilon/4)$. Then F is a non-empty closed subset of A and $i_{A,X}(F) = cl_X F \in \mathcal{W}_X(x,R)$. By the continuity of the map $i_{A,X}$, it follows that there exists a neighbourhood $\mathbf{V}$ of F from the base of the $\mathcal{W}^+$-hypertopology on $CL(A)$ such that $i_{A,X}(\mathbf{V}) \subseteq \mathcal{W}_X(x,R)$. The neighbourhood $\mathbf{V}$ has the form $\cap_{k=1}^m \mathcal{W}_A(a_k, r_k)$ for some $a_k \in A$ and $r_k \geqq 0$ $(k = 1, \dots, m)$. Since $F \in \mathbf{V}$, for every $k = 1, \dots, m$, there exists $R_k > r_k$ such that $F \cap B_X(a_k, R_k) = \varnothing$, i.e.
$B_X(a_k, R_k) \subseteq B_X(x, R+\varepsilon/4)$ (1).

Since $i_{A,X}(\mathbf{V}) \subseteq \mathcal{W}_X(x,R)$, we obtain that $z \in A \setminus \cup_{k=1}^m B_X(a_k,R_k)$ implies $z \notin B_X(x,R)$. Consequently $A \cap B_X(x,R) \subseteq \cup_{k=1}^m B_X(a_k,R_k)$. But, obviously, $D = A \cap B_X(x,R)$. Therefore, $D \subseteq \cup_{k=1}^m B_X(a_k,R_k)$ (2).

We are going to show that $R_k < \varepsilon/2$ for every $k = 1, \dots, m$. Let k be fixed. We set $y = a_k + (R_k - \delta)\frac{(a_k - x)}{\|a_k - x\|}$, where δ is such that $0 < \delta < R_k$. Then $y \in B_X(a_k, R_k)$ and, by (1), we obtain that $y \in B_X(x, R+\varepsilon/4)$. Hence $R+\varepsilon/4 > \|y-x\| = \|a_k - x\| + R_k - \delta$. From $a_k \in A$, it follows that $\|a_k - x\| \geqq r$, and therefore $R + \varepsilon/4 > r + R_k - \delta$. Consequently $R_k - \delta < (R-r) + \varepsilon/4$ for every $\delta \in (0, R_k)$. Hence, $R_k \leqq (R-r) + \varepsilon/4 < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$.

From $R_k < \varepsilon/2$ (for each $k = 1, \dots, m$) and (2), we obtain that $D \subseteq \cup_{k=1}^m B_X(a_k, \varepsilon/2)$. Obviously, there is no loss of generality in assuming that $D \cap B_X(a_k, \varepsilon/2) \neq \varnothing$ for every $k = 1, \dots, m$. For every $k = 1, \dots, m$ we choose a point $c_k \in D \cap B_X(a_k, \varepsilon/2)$ and set $C = \{c_1, \dots, c_m\}$. Then C is a finite ε-dense subset of D. Thus, the proof of lemma 7.6 is completed.□

Now, we are going to show that the metric d is B-TB. Let $B_X(x_0, R_0)$ be an arbitrary open d-ball and $\varepsilon > 0$. We choose $m \in \mathbb{N}$ such that $\frac{R_0}{m} < \frac{\varepsilon}{5}$, i.e. $\frac{5R_0}{\varepsilon} < m$. For every $k = 1, \dots, m$ we set $D_k = \{y \in X : \frac{R_0}{m}(k-1) \leqq d(x_0, y) < \frac{R_0}{m}k\}$. For every $k = 2, \dots, m$, the set D_k satisfies the conditions of lemma 7.6. Therefore, for every $k = 2, \dots, m$, there exists a finite ε-dense subset of D_k. Obviously, the set $\{x_0\}$ is ε-dense in D_1. Then, from $B_X(x_0, R_0) = \cup_{k=1}^m D_k$, we conclude that there exists a finite ε-dense subset of $B_X(x_0, R_0)$. ■

Corollary 7.7. $\mathcal{HS}(T, Lin.norm.sp.) = \mathcal{COM}(T, Lin.norm.sp.) =$
$=$ { *finite dimensional spaces* } *for* $T \in \{\mathcal{W}, \mathcal{W}^+\}$.

Example 7.8. We will costruct a metric space (X, d) such that the metric d is neither B-TB, nor ultrametric, but, for every subspace A of X, the map $i_{A,X} : (CL(A), \mathcal{W}^+(d|A \times A)) \longrightarrow (CL(X), \mathcal{W}^+(d))$ is continuous.

Let (X_1, d_1) and (X_2, d_2) be bounded metric spaces such that d_1 is an ultrametric which is not B-TB and the metric d_2 is B-TB, but not ultrametric. Then we set $X = X_1 \oplus X_2$ and define a metric d on X by the following way:

$$d(x, y) = \begin{cases} d_1(x, y) \text{ if } x, y \in X_1 \\ d_2(x, y) \text{ if } x, y \in X_2 \\ \quad M \text{ in all other cases,} \end{cases}$$

where M is a constant which bounds both d_1 and d_2. By proposition 7.3, (X_1, d_1) and (X_2, d_2) belong to the class $\mathcal{HS}(\mathcal{W}^+, Metr.sp.)$. Then, it is easy to check that the sum (X, d) belongs to $\mathcal{HS}(\mathcal{W}^+, Metr.sp.)$ too, but, evidently, the metric d is neither B-TB, nor ultrametric. Therefore, it suffices to find spaces (X_1, d_1) and (X_2, d_2) with the above properties.

Obviously, the interval [0,1] endowed with the standard Euclidean metric has all properties of (X_2, d_2). Now we will construct the space (X_1, d_1). Let X_1 be the set of all formal power series of an argument x over the ring $\mathbb{Z}$ of the integers. For arbitrary elements $f = \sum_{k=0}^{\infty} a_k x^k$ and $g = \sum_{k=0}^{\infty} b_k x^k$ of X_1 we put: $f \equiv g \pmod{x^s}$ iff for every $k = 0, \ldots, s-1$ we have that $a_k = b_k$. For every non-zero element f of X_1 we put: $\nu(f) = \max\{s : f \equiv 0 \pmod{x^s}\}$. When $f, g \in X_1$, we put :

$$d_1(f, g) = \begin{cases} \rho^{\nu(f-g)} \text{ if } f \text{ is different from } g \\ \quad 0 \text{ if } f \text{ coincides with } g, \end{cases}$$

where ρ is a fixed real number between 0 and 1.

Let $f = \sum_{k=0}^{\infty} a_k x^k$, $n \in \mathbb{N}$, and $\varepsilon = \rho^n$. We will show that the open ball $B(f, \rho^{n-1})$ has not a finite ε-dense subset. Let's assume that $\{h_k\}_{k=1}^m$ is a ε-dense subset of $B(f, \rho^{n-1})$. Let a be an integer which is different from the coefficient of x^n in h_k for every $k = 1, \ldots, m$. We set $h = (\sum_{k=0}^{n-1} a_k x^k) + a x^n$. Then $f - h \equiv 0 \pmod{x^n}$ and $h_k - h \not\equiv 0 \pmod{x^{n+1}}$ for every $k = 1, \ldots, m$. Therefore $h \in B(f, \rho^{n-1})$, but $h \notin B(h_k, \rho^n)$ for every $k = 1, \ldots, m$. Thus we obtain a contradiction, which shows that $B(f, \rho^{n-1})$ is not totally bounded. Hence, d_1 is not B-TB.■

8. Proximal ball hypertopology $\mathcal{B}\delta$

Definition 8.1. Let (X, d) be a metric space. *The lower proximal ball hypertopology* $\mathcal{B}\delta^-$ on $CL(X)$ coincides, by definition, with the lower Vietoris hypertopology $\mathcal{V}^-$ on $CL(X)$. *The upper proximal ball hypertopology* $\mathcal{B}\delta^+$ on $CL(X)$ has for a subbase the family $\{(X \setminus \overline{B}_X(x, \varepsilon))^{++} : x \in X \text{ and } \varepsilon \in \mathbb{R}\}$. *The proximal ball hypertopology* $\mathcal{B}\delta$ on $CL(X)$ is the supremum of $\mathcal{B}\delta^-$ and $\mathcal{B}\delta^+$.

Proposition 8.2 ([4, (15.5)]). *Let (X, d) be a metric space, and let the metric d be B-TB. Then the proximal ball hypertopology $\mathcal{B}\delta$ on $CL(X)$ and the Wijsman hypertopology $\mathcal{W}$ on $CL(X)$ coincide.*

Proposition 8.3. *Let (X, d) be a metric space, and let the metric d be B-TB. Then (X, d) belongs to the class $\mathcal{COM}(T, Metr.sp.)$ for $T \in \{\mathcal{B}\delta, \mathcal{B}\delta^+\}$.*

Proof. Propositions 8.2, 7.2 and 7.4 imply that $(X, d) \in \mathcal{COM}(\mathcal{B}\delta, Metr.sp.)$. Then, by lemma 2.1, we obtain that $(X, d) \in \mathcal{COM}(\mathcal{B}\delta^+, Metr.sp.)$.■

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