Math 242, Fall 2003 Test 2 Name:....................

1. (14 points) Given the fact that $y_1(x) = x^2$ is a solution of the differential equation

$$x^2y'' + 2xy' - 6y = 0$$

use the method of reduction of order to find another "essentially different" solution of the given equation.

2 $y_2(x) = u(x)\, y_1(x) = x^2 \cdot u$

1 $y_2' = 2xu + x^2u'$

2 $y_2'' = 2u + 2xu' + 2xu' + x^2u'' = 2u + 4xu' + x^2u''$

So $x^2y'' + 2xy' - 6y = 0$ becomes

1 $x^2(2u + 4xu' + x^2u'') + 2x(2xu + x^2u') - 6x^2u = 0$

1 $\cancel{2x^2u} + 4x^3u' + x^4u'' + \cancel{4x^2u} + 2x^3u' - \cancel{6x^2u} = 0$

$x^4u'' + 6x^3u' = 0 \quad | \div x^3$

1 $xu'' + 6u' = 0$

Let $v = u'$. Then

$xv' + 6v = 0$

$xv' = -6v$

1 $\frac{v'}{v} = -\frac{6}{x}$

3 $\int \frac{v'}{v}\,dx = -6\int \frac{1}{x}\,dx$

$\ln|v| = -6\ln|x| + C$ [illegible]

$e^{\ln|v|} = e^{\ln|x|^{-6}} \cdot e^C$

$|v| = \frac{C_1}{x^6}$

$v = \frac{C_1}{x^6}$ [illegible]

$u' = v = \frac{C_1}{x^6}$

1 So $u = \int u'\,dx = \int \frac{C_1}{x^6}\,dx = C_1\int x^{-6}\,dx =$

$= \frac{C_1x^{-5}}{-5} + C = -\frac{C_1}{5x^5} + C$

pick $C = 0$

$C_1 = -5$

Thus $u(x) = \frac{1}{x^5}$

and $y_2(x) = \frac{1}{x^5} \cdot x^2 = \boxed{\frac{1}{x^3}}$

1

2. (14 points) Solve the homogeneous differential equation:

$$\begin{cases} y'' - 4y' + 5y = 0 \\ y(0) = 2, \ y'(0) = 2. \end{cases}$$

2 $t^2 - 4t + 5 = 0$

2 $t = \dfrac{4 \pm \sqrt{16 - 4\cdot 5}}{2} = \dfrac{4 \pm \sqrt{-4}}{2} = \underbrace{2}_{\alpha} \pm \underbrace{1}_{\beta} i$

2 $y(x) = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x$

$2 = y(0) = C_1 \cdot 1 \cdot 1 + C_2 \cdot 1 \cdot 0$

So $\boxed{C_1 = 2}$ 2

2 $y'(x) = 2C_1 e^{2x} \cos x - C_1 e^{2x} \sin x + 2C_2 e^{2x} \sin x + C_2 e^{2x} \cos x$

$2 = y'(0) = 2C_1 \cdot 1 \cdot 1 - C_1 \cdot 1 \cdot 0 + 2C_2 \cdot 1 \cdot 0 + C_2 \cdot 1 \cdot 1 = 2C_1 + C_2$

$C_2 = 2 - 2 \cdot 2 = -2$ 2

$y(x) = 2e^{2x} \cos x - 2e^{2x} \sin x = \boxed{2e^{2x}(\cos x - \sin x)}$

3. (14 points) Solve the initial value problem:

$$\begin{cases} y'' - 5y' + 6y = -36x + 20\sin x \\ y(0) = 1, \ y'(0) = -1. \end{cases}$$

1) $t^2 - 5t + 6 = 0$

1 $(t-2)(t-3) = 0$

1 $t_1 = 2 \quad t_2 = 3$

2 $y_{hom} = C_1 e^{2x} + C_2 e^{3x}$

2) $y_{p_1}(x) = Ax + B$ 1

$\left.\begin{array}{l} y_{p_1}' = A \\ y_{p_1}'' = 0 \end{array}\right\}$ 0.5

0.5 $0 - 5 \cdot A + 6(Ax + B) = -36x$

$-5A + 6Ax + 6B = -36x$

0.5 $\left| \begin{array}{l} 6A = -36 \\ -5A + 6B = 0 \end{array} \right.$ $\quad A = -6$ $\quad B = \frac{5}{6}A = -5$ 0.5

$y_{p_1}(x) = -6x - 5$ 0.5

3) $y_{p_2}(x) = C\sin x + D\cos x$ 1

$\left.\begin{array}{l} y_{p_2}' = C\cos x - D\sin x \\ y_{p_2}'' = -C\sin x - D\cos x \end{array}\right\}$ 0.5

0.5 $-C\sin x - D\cos x - 5C\cos x + 5D\sin x + 6C\sin x + 6D\cos x$

$= 20\sin x$

0.5 $\left| \begin{array}{l} -C + 5D + 6C = 20 \\ -D - 5C + 6D = 0 \end{array} \right.$ i.e. $\left| \begin{array}{l} 5D + 5C = 20 \\ 5D - 5C = 0 \end{array} \right.$

$D = C$ and $10C = 20$

So $C = D = 2$ 0.5

$y_{p_2}(x) = 2\sin x + 2\cos x$ 0.5

$y(x) = C_1 e^{2x} + C_2 e^{3x} - 6x - 5 + 2(\sin x + \cos x)$

1 $y'(x) = 2C_1 e^{2x} + 3C_2 e^{3x} - 6 + 2(\cos x - \sin x)$

$1 = y(0) = C_1 + C_2 - 5 + 2(0+1)$

$-1 = y'(0) = 2C_1 + 3C_2 - 6 + 2(1-0)$

$\left| \begin{array}{l} C_1 + C_2 = 4 \\ 2C_1 + 3C_2 = 3 \end{array} \right.$

$C_1 = 4 - C_2$

$8 - 2C_2 + 3C_2 = 3$

$C_2 = 3 - 8 = -5$ 1

$C_1 = 4 - C_2 = 4 - (-5) = 9$ 1

$$\boxed{y(x) = 9e^{2x} - 5e^{3x} - 6x - 5 + 2(\sin x + \cos x)}$$

4. (8 points each) For each of the equations below:

(i) Find all solutions of the homogeneous part of the equation;

(ii) Set the general form of the particular solution. You are not required to calculate the coefficients and complete the solution of the problem.

$$\text{(a) } y'' + 4y' + 4y = (3+x)e^{-2x}$$

(i) $t^2 + 4t + 4 = 0$ 1

$(t+2)^2 = 0$ 1

$t = -2$ mult. 2 1

$y_{hom}(x) = c_1 e^{-2x} + c_2 x e^{-2x}$ 1

(ii) $y_p(x) = (Ax+B)e^{-2x} \cdot x^2$ 1

1 1 1

$$\text{(b) } y''' + 8y = 8e^{-2x}\cos 2x.$$

(i) $t^3 + 8 = 0$ 1

$(t+2)(t^2 - 2t + 4) = 0$

$t = -2$ 1 $\quad t_{1,2} = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \frac{\sqrt{-12}}{2} = 1 \pm \sqrt{3}i$ 1

$y_{hon}(x) = C_1 e^{-2x} + c_2 e^x \sin(\sqrt{3}x) + c_3 e^x \cos(\sqrt{3}x)$ 1

(ii) $y_p =$ [redacted] $e^{-2x}(A\cos 2x + B\sin 2x)$ 1

1 1 1

5. (14 points) Solve the initial value problem:

$$\begin{cases} y'' + y = f(t) \\ y(0) = 0, \ y'(0) = 1, \end{cases}$$

where

$$f(t) = \begin{cases} \sin t, & \text{if } \ 0 \le t < \frac{\pi}{2} \\ \cos 2t, & \text{if } \ \frac{\pi}{2} \le t. \end{cases}$$

$$f(t) = \left(1 - \mathcal{U}\left(t - \tfrac{\pi}{2}\right)\right)\sin t + \mathcal{U}\left(t - \tfrac{\pi}{2}\right)\cos 2t =$$

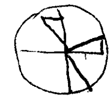
1

$$= \sin t + \mathcal{U}\left(t-\tfrac{\pi}{2}\right)\left(\cos 2t - \sin t\right)$$

$$0.5 \quad s^2 Y(s) - s\overset{0}{y(0)} - \overset{1}{y'(0)} + Y(s) \overset{0.5}{=} \mathcal{L}\{\sin t\} + \mathcal{L}\left\{\mathcal{U}\left(t - \tfrac{\pi}{2}\right)(\cos 2t - \sin t)\right\}$$

T2 direct $a = \frac{\pi}{2}$

$$Y(s)(s^2+1) \overset{0.5}{-} 1 = \frac{1}{s^2+1}^{0.5} + e^{-\frac{\pi}{2}s}\overset{0.5}{\mathcal{L}}\left\{\cos 2\left(t+\tfrac{\pi}{2}\right) - \sin\left(t + \tfrac{\pi}{2}\right)\right\}$$



$$Y(s)(s^2+1) \overset{0.5}{=} 1 + \frac{1}{s^2+1} + e^{-\frac{\pi}{2}s}\overset{0.5}{\mathcal{L}}\left\{-\cos 2t - \cos t\right\}$$

$$Y(s)(s^2+1) = 1 + \frac{1}{s^2+1} + e^{-\frac{\pi}{2}s}\overset{0.5}{\left(-\frac{s}{s^2+4} - \frac{s}{s^2+1}\right)}$$

$$0.5 \quad Y(s) = \frac{1}{s^2+1} + \frac{1}{(s^2+1)^2} - e^{-\frac{\pi}{2}s}\left(\frac{s}{(s^2+1)(s^2+4)} + \frac{s}{(s^2+1)^2}\right)$$

T2 inverse $a = \frac{\pi}{2}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} - \mathcal{U}\left(t-\tfrac{\pi}{2}\right)\left(\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{s}{(s^2+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}\right) =$$

#25, k = 1   #22, k = 1

$$= \sin t \overset{0.5}{+} \frac{1}{2}\left(\sin t - t\cos t\right)$$

$$- \mathcal{U}\left(t - \tfrac{\pi}{2}\right)\left(\frac{1}{3}\cos\left(t - \tfrac{\pi}{2}\right) \overset{0.5}{-} \frac{1}{3}\cos 2\left(t - \tfrac{\pi}{2}\right) + \frac{1}{2}\left(t - \tfrac{\pi}{2}\right)\sin\left(t - \tfrac{\pi}{2}\right)\right) =$$

$$= \frac{3}{2}\sin t - \frac{1}{2}t\cos t - \mathcal{U}\left(t - \tfrac{\pi}{2}\right)\left(\frac{1}{3}\sin t + \frac{1}{3}\cos 2t - \frac{1}{2}\left(t - \tfrac{\pi}{2}\right)\cos t\right)$$

$$\boxed{y(t) = \frac{3}{2}\sin t - \frac{1}{2}t\cos t - \mathcal{U}\left(t - \tfrac{\pi}{2}\right)\left(\frac{1}{3}\sin t + \frac{1}{3}\cos 2t - \frac{1}{2}\left(t - \tfrac{\pi}{2}\right)\cos t\right)}$$

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \quad 1$$

$$s = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$s = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + Cs + D \quad 1$$

$A + C = 0$ $C = -A$

$B + D = 0$ $D = -B$

$4A + C = 1$ $4A - A = 1$ $\boxed{A = \frac{1}{3}}$ .5

$4B + D = 0$ $4B - B = 0$ $\boxed{B = 0}$ .5

$\boxed{C = -\frac{1}{3}}$ .5 $\boxed{D = 0}$ .5

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{s}{(s^2+1)} - \frac{1}{3}\frac{s}{(s^2+4)}$$

6. (14 points) Solve the initial value problem:

$$\begin{cases} y'' - 2y' + y = t^3 e^t \\ y(0) = 0, \ \ y'(0) = 1. \end{cases}$$

$$s^2 Y(s) - s\overset{0}{y(0)} - \overset{1}{y'(0)} - 2\left(sY(s) - \overset{0}{y(0)}\right) + Y(s) = \mathcal{L}\{e^t \cdot t^3\}$$

T1 direct $a=1$

1 $Y(s)(s^2 - 2s + 1) - 1 = \mathcal{L}\{t^3\}(s-1)$ 1

1 $Y(s)(s-1)^2 = 1 + \dfrac{3!}{(s-1)^4}$ 1

1 $Y(s) = \dfrac{1}{(s-1)^2} + \dfrac{6}{(s-1)^6}$

T1 inverse, $a=1$ 1.5 1.5

1 $y(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2}\right\} + 6\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^6}\right\} = e^t \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} + 6e^t \mathcal{L}^{-1}\left\{\dfrac{1}{s^6}\right\} =$

$= e^t\left(t + 6 \cdot \dfrac{1}{5!} t^5\right) = e^t\left(t + \dfrac{1}{20} t^5\right)$ 1 1

$$\boxed{y(t) = e^t t\left(1 + \frac{1}{20} t^4\right)}$$

7. (14 points) Use the Laplace transform to solve the integral equation:

$$f(t) + \int_0^t f(\tau)\, d\tau = 2t.$$

1 $\mathcal{L}\{f(t)\} + \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = 2\mathcal{L}\{t\}$

1 $F(s) + \dfrac{1}{s} F(s) = 2 \cdot \dfrac{1}{s^2}$ 2 1

1 $F(s)\left(1 + \dfrac{1}{s}\right) = \dfrac{2}{s^2}$ 1.5

1 $F(s)(s+1) = \dfrac{2}{s}$

1 $F(s) = \dfrac{2}{s(s+1)}$

1 $\dfrac{2}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1} = \dfrac{2}{s} - \dfrac{2}{s+1}$

1 $2 = As + A + Bs$

$\begin{cases} A = 2 \\ B = -A = -2 \end{cases}$ 1

1 $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s}\right\} - \mathcal{L}^{-1}\left\{\dfrac{2}{s+1}\right\} = 2 - 2e^{-t}$ 1 1

$$\boxed{f(t) = 2(1 - e^{-t})}$$

8. BONUS. (10 points or nothing) Find all solutions of the differential equation:

$$4y'' - 4y' + y = e^{\frac{x}{2}}\sqrt{1-x^2}$$

$$y'' - y' + \frac{1}{4}y = \frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2}$$

1) $t^2 - t + \frac{1}{4} = 0$

$\left(t - \frac{1}{2}\right)^2 = 0$

$t = \frac{1}{2}$ mult. 2

$$y_{hom} = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$$

2) $y_p = u_1(x)\underbrace{e^{\frac{x}{2}}}_{y_1} + u_2(x)\underbrace{x e^{\frac{x}{2}}}_{y_2}$

$A = u_1'$
$B = u_2'$

$$\begin{cases} e^{\frac{x}{2}} \cdot A + x e^{\frac{x}{2}} B = 0 & \Big| \div e^{\frac{x}{2}} \\ \frac{1}{2} e^{\frac{x}{2}} A + e^{\frac{x}{2}}\left(1 + \frac{x}{2}\right) B = \frac{1}{4} e^{\frac{x}{2}} \sqrt{1-x^2} & \Big| \div \left(e^{\frac{x}{2}}\right) \end{cases}$$

$$\begin{cases} A = -xB \\ \frac{1}{2}A + \left(1 + \frac{x}{2}\right) B = \frac{1}{4}\sqrt{1-x^2} \end{cases}$$

$\frac{1}{2}\cancel{(-x)B} + B + \cancel{\frac{x}{2}B} = \frac{1}{4}\sqrt{1-x^2}$

$B = \frac{1}{4}\sqrt{1-x^2}$

$A = -xB = -\frac{x}{4}\sqrt{1-x^2}$

$v = 1 - x^2$
$dv = -2x\,dx$

$u_1(x) = \int A\,dx = \int -\frac{x}{4}\sqrt{1-x^2}\,dx =$

$= \frac{1}{8}\int \sqrt{1-x^2}\,(-2x)\,dx = \frac{1}{8}\int v^{\frac{1}{2}}\,dv =$

$= \frac{1}{8} \cdot \frac{2v^{\frac{3}{2}}}{3} = \frac{1}{12}(1-x^2)^{\frac{3}{2}} + C$ pick $C = 0$

$u_2(x) = \int B\,dx = \frac{1}{4}\int \sqrt{1-x^2}\,dx = \boxed{\begin{matrix}\#85 \\ a = 1\end{matrix}}$

$= \frac{1}{4}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x\right) + C$ pick $C = 0$

$$y_p(x) = \frac{1}{12}(1-x^2)^{\frac{3}{2}} e^{\frac{x}{2}} + \frac{1}{8}\left(x\sqrt{1-x^2} + \arcsin x\right) x e^{\frac{x}{2}}$$

$$\boxed{y(x) = \left(c_1 + c_2 x + \frac{1}{12}(1-x^2)^{\frac{3}{2}} + \frac{1}{8}x^2\sqrt{1-x^2} + \frac{1}{8}x \arcsin x\right) e^{\frac{x}{2}}}$$