

Math 242, Fall 2003 Test 1 Name:.....Make-up.....

1. (10 points) Determine whether the theorem for existence of a unique solution guarantees that the differential equation

$$y' = 2y\sqrt{x^2 - 4}$$

possesses a unique solution passing through the point (2, 1). Show all your work and explain clearly what motivates your answer.

$f(x, y) = 2y\sqrt{x^2 - 4}$ defined & continuous for $x \leq -2$ and $x \geq 2$
 $\frac{\partial f}{\partial y} = 2\sqrt{x^2 - 4}$ ———

but any rectangle around (2, 1) would contain points with $-2 < x < 2$ so we can not apply the theorem

2. (15 points) Solve the initial value problem:

$$\begin{cases} e^{-3x^2} y' - 4x^3 y = 2x^3 + 2x^3 y^2 \\ y(0) = -10 \end{cases}$$

$$e^{-3x^2} y' = 2x^3 (y^2 + 2y + 1)$$

$$y' = 2x^3 e^{3x^2} (y+1)^2$$

$$\int \frac{y'}{(y+1)^2} dx = \int 2x^3 e^{3x^2} dx + C$$

$$v = 3x^2 \quad dv = 6x dx \quad x^2 = \frac{v}{3}$$

$w = y+1$
 $dw = dy$

$$\int \frac{dw}{w^2} = \frac{1}{3} \int x^2 e^{3x^2} \cdot 6x dx + C$$

$$\int \frac{dw}{w^2} = \frac{1}{3} \int \frac{v}{3} e^v dv + C$$

$$\frac{-1}{w} = \frac{1}{9} \int v e^v dv + C$$

$$\frac{-1}{w} = \frac{1}{9} (v e^v - \int e^v dv) + C$$

$$\frac{-1}{w} = \frac{1}{9} e^v (v-1) + C$$

$$\frac{-1}{y+1} = \frac{1}{9} e^{3x^2} (3x^2-1) + C$$

$$\frac{z}{e} = v \quad \frac{1}{e} = e^v \\ dz = dv \quad dt = e^v$$

$$y+1 = \frac{-1}{\frac{1}{9} e^{3x^2} (3x^2-1) + C}$$

$$y = \frac{-1}{\frac{1}{9} e^{3x^2} (3x^2-1) + C} - 1$$

$$-10 = \frac{-1}{\frac{1}{9} e^0 (0-1) + C} - 1$$

$$-9 = \frac{-1}{-\frac{1}{9} + C}$$

$$C - \frac{1}{9} = \frac{1}{9}$$

$$C = \frac{2}{9}$$

$$y = \frac{-1}{\frac{1}{9} e^{3x^2} (3x^2-1) + \frac{2}{9}} - 1 =$$

$$= \frac{-9}{e^{3x^2} (3x^2-1) + 2} - 1$$

3. (15 points) Solve the initial value problem:

$$\begin{cases} xy' - y = x^2 \sin x \\ y(\frac{\pi}{2}) = \pi \end{cases}$$

$$y' - \frac{1}{x}y = x \sin x$$

$$1) y' - \frac{1}{x}y = 0$$

$$y' = \frac{1}{x}y \quad x > 0$$

$$\int \frac{y'}{y} dx = \int \frac{1}{x} dx$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$|y| = x \cdot C_1$$

$$y = \cancel{cx} \cdot Cx$$

$$2) y_p = u(x) \cdot x$$

$$y_p' = u' \cdot x + u \cdot 1$$

$$u'x + u - \frac{1}{x} \cdot u \cdot x = x \sin x$$

$$u' = \sin x$$

$$u = -\cos x + C$$

$$u = -\cos x$$

$$y_p = -x \cdot \cos x$$

pick $C=0$

$$3) y = Cx - x \cos x$$

$$\pi = C \cdot \frac{\pi}{2} - \frac{\pi}{2} \cdot \cos\left(\frac{\pi}{2}\right)$$

$$C = 2$$

$$y = 2x - x \cos x$$

4. (15 points) Solve the initial value problem:

$$\begin{cases} (4xy^2 - 3) + (4x^2y + 2)y' = 0 \\ y(1) = 1 \end{cases}$$

$$\frac{\partial M}{\partial y} = 4x \cdot 2y = 8xy$$

$$\frac{\partial N}{\partial x} = 4 \cdot 2x \cdot y = 8xy$$

$$F = \int \frac{\partial F}{\partial x} dx = \int M(x,y) dx = \int (4xy^2 - 3) dx + g(y) =$$

$$= 4y^2 \cdot \frac{x^2}{2} - 3x + g(y) = 2x^2y^2 - 3x + g(y)$$

$$\frac{\partial F}{\partial y} = N(x,y)$$

$$2x^2 \cdot 2y - 0 + g'(y) = 4x^2y + 2$$

$$g'(y) = 2$$

$$g(y) = 2y + C$$

$$F = 2x^2y^2 - 3x + 2y + C = 0$$

$$2 \cdot 1 \cdot 1 - 3 \cdot 1 + 2 \cdot 1 + C = 0$$

$$2 - 3 + 2 + C = 0$$

$$C = -1$$

$$\boxed{2x^2y^2 - 3x + 2y - 1 = 0}$$

5. (15 points) Solve the initial value problem using an appropriate substitution:

$$\begin{cases} y^{\frac{1}{2}} y' + y^{\frac{3}{2}} = 1 \\ y(0) = 4 \end{cases}$$

$$y' + y = y^{-\frac{1}{2}}$$

$$u = y^{1 - (-\frac{1}{2})} = y^{\frac{3}{2}}$$

$$u' = \frac{3}{2} \cdot y^{\frac{1}{2}} \cdot y'$$

$$y' = \frac{2}{3} \frac{u'}{y^{\frac{1}{2}}}$$

$$\frac{2}{3} \cdot \frac{u'}{y^{\frac{1}{2}}} + y = y^{-\frac{1}{2}} \quad | \cdot \frac{3}{2} y^{\frac{1}{2}}$$

$$u' + \frac{3}{2} y^{\frac{3}{2}} = \frac{3}{2}$$

$$u' + \frac{3}{2} u = \frac{3}{2}$$

$$u' = \frac{3}{2}(1-u)$$

$$\int \frac{u'}{1-u} dx = \int \frac{3}{2} dx$$

$$u > 1 \quad \text{since } u = y^{\frac{3}{2}} = 4^{\frac{3}{2}} = 8 > 1$$

$$\int \frac{dy}{1-u} = \frac{3}{2} x + C$$

$$y^{\frac{3}{2}} - 1 = 7e^{-\frac{3}{2}x}$$

$$- \ln|1-u| = \frac{3}{2}x + C$$

$$\boxed{y^{\frac{3}{2}} = 1 + 7e^{-\frac{3}{2}x}}$$

$$e^{\ln|1-u|} = e^{-\frac{3}{2}x - C}$$

$$|1-u| = e^{-\frac{3}{2}x} \cdot C$$

$$|1 - y^{\frac{3}{2}}| = Ce^{-\frac{3}{2}x}$$

$$|1 - 8| = C \cdot 1$$

$$C = 7$$

6. (15 points) Solve the initial value problem using an appropriate substitution:

$$\begin{cases} y^2 + x\sqrt{x^2 + y^2} - xy y' = 0 \\ y(1) = \sqrt{3} \end{cases}$$

$$u = \frac{y}{x} \quad y = ux \quad y' = u'x + u \quad \boxed{x > 0}$$

$$u^2 x^2 + x\sqrt{x^2 + u^2 x^2} - xux \cdot (u'x + u) = 0$$

$$x^2 u^2 + x^2 \sqrt{1 + u^2} - x^2 u (u'x + u) = 0 \quad | \div x^2 \quad \boxed{x > 0}$$

$$u^2 + \sqrt{1 + u^2} - ux \cdot u' - u^2 = 0$$

$$ux u' = \sqrt{1 + u^2}$$

$$\int \frac{uu'}{\sqrt{1+u^2}} dx = \int \frac{1}{x} dx$$

$$v = 1 + u^2 \quad dv = 2u du \quad \frac{1}{2} \int \frac{2u du}{\sqrt{1+u^2}} = \ln|x| + C$$

$$\frac{\sqrt{x^2 + y^2}}{x} = \ln x + 2$$

$$\frac{1}{2} \int \frac{dv}{\sqrt{v}} = \ln|x| + C$$

$$\frac{1}{2} \int v^{-1/2} dv = \ln|x| + C$$

$$\frac{1}{2} \cdot \frac{v^{1/2}}{1/2} = \ln|x| + C$$

$$(1 + u^2)^{1/2} = \ln|x| + C$$

$$\sqrt{1 + \frac{y^2}{x^2}} = \ln|x| + C$$

$$\sqrt{1 + \frac{3}{1}} = \frac{\ln 1}{\sqrt{0}} + C$$

$$C = 2$$

$$\boxed{\sqrt{x^2 + y^2} = x(\ln x + 2)}$$

7. (15 points) Solve the initial value problem by finding an appropriate integrating factor, if necessary:

$$\begin{cases} (\sin x - \sin x \cos x \sin y) + \cos^2 x \cos y y' = 0 \\ y(0) = \frac{\pi}{2} \end{cases}$$

$$\frac{\partial M}{\partial y} = -\sin x \cos x \cdot \cos y \quad \neq \quad \frac{\partial N}{\partial x} = 2 \cos x \cdot (-\sin x) \cos y = -2 \sin x \cos x \cdot \cos y$$

$$(\sin x - \sin x \cos x \sin y) \mu(x) + \cos^2 x \cos y \mu(x) \cdot y' = 0$$

$$\frac{\partial M}{\partial y} = -\sin x \cos x \cdot \cos y \cdot \mu$$

$$\frac{\partial N}{\partial x} = 2 \cos x (-\sin x) \cos y \cdot \mu + \cos^2 x \cdot \cos y \cdot \mu'$$

$$\cos^2 x \cos y \mu' + 2 \sin x \cos x \cos y \cdot \mu = -\sin x \cos x \cos y \mu$$

$$\cos^2 x \cos y \cdot \mu' = -\sin x \cos x \cos y \cdot \mu$$

$$\frac{\mu'}{\mu} = \frac{\sin x \cdot \cancel{\cos x} \cdot \cancel{\cos y}}{\cos^2 x \cdot \cancel{\cos y}} = \frac{\sin x}{\cos x}$$

$$\int \frac{\mu'}{\mu} dx = \int \frac{\sin x}{\cos x} dx \quad \begin{matrix} u = \cos x \\ du = -\sin x \end{matrix}$$

$$\ln |\mu| = - \int \frac{du}{u} = -\ln |u| + C = -\ln |\cos x| + C \quad \text{pick } C=0$$

$$e^{\ln |\mu|} = e^{\ln |\cos x|^{-1}}$$

$$|\mu| = \frac{1}{|\cos x|}$$

$$\mu = \frac{1}{\cos x}$$

$$\left(\frac{\sin x}{\cos x} - \sin x \sin y \right) + \cos x \cdot \cos y \cdot y' = 0$$

cont.

#7 (cont.)

$$F = \int \frac{\partial F}{\partial y} dy = \int \cos x \cdot \cos y dy + g(x) = \cos x \cdot \sin y + g(x)$$

$$M(x, y) = \frac{\partial F}{\partial x}$$

$$\frac{\sin x}{\cos x} - \cancel{\sin x \cdot \cos y} = (-\cancel{\sin x}) \sin y + g'(x)$$

$$g'(x) = \frac{\sin x}{\cos x}$$

$$u = \cos x$$

$$du = -\sin x$$

$$\cos x > 0$$

$$g(x) = \int \frac{\sin x}{\cos x} dx = - \int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

$$\cos x \cdot \sin y - \ln(\cos x) + C = 0$$

$$\cos 0 \cdot \sin\left(\frac{\pi}{2}\right) - \ln(\cos 0) + C = 0$$

$$1 - 0 + C = 0$$

$$C = -1$$

$$\cos x \cdot \sin y - \ln(\cos x) - 1 = 0$$