1

$$y'' + 6y' + 5y = t - 1 - t\,\mathcal{U}(t-1) \qquad y(0) = 1 \quad y'(0) = -1$$

$$s^2 Y(s) - s\cdot \underset{1}{\underset{\shortparallel}{y(0)}} - \underset{-1}{\underset{\shortparallel}{y'(0)}} + 6\left(s\,Y(s) - \underset{1}{\underset{\shortparallel}{y(0)}}\right) + 5\,Y(s) = \frac{1}{s^2} - \frac{1}{s} - e^{-s}\frac{1}{(s+1)^2}$$

$$Y(s)\left(s^2 + 6s + 5\right) - s + 1 - 6 = \frac{1-s}{s^2} - e^{-s}\frac{1}{(s+1)^2}$$

$$Y(s)\,(s+5)(s+1) = s + 5 + \frac{1-s}{s^2} - e^{-s}\cdot\frac{1}{(s+1)^2}$$

$$Y(s) = \frac{1}{s+1} + \frac{1-s}{s^2(s+1)(s+5)} - e^{-s}\cdot\frac{1}{(s+1)^3(s+5)}$$

$$\frac{1-s}{s^2(s+1)(s+5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+5} = -\frac{11}{25}\cdot\frac{1}{s} + \frac{1}{5}\cdot\frac{1}{s^2} + \frac{1}{2}\cdot\frac{1}{(s+1)} - \frac{3}{50}\cdot\frac{1}{(s+5)}$$

$$1 - s = As(s+1)(s+5) + B(s+1)(s+5) + Cs^2(s+5) + Ds^2(s+1)$$

if $s = 0$: $1 = 0 + 5B + 0 + 0$ So $B = \frac{1}{5}$

if $s = -1$: $2 = 0 + 0 + C\cdot 4 + 0$ So $C = \frac{1}{2}$

if $s = -5$: $6 = 0 + 0 + 0 + D\cdot 25(-4)$ So $D = -\frac{6}{100} = -\frac{3}{50}$

Comparing the coefficients in front of s^3 we get:

$$0 = A + C + D \qquad \text{So} \qquad A = -C - D = -\frac{1}{2} + \frac{3}{50} = \frac{-50+6}{100} = -\frac{44}{100} = -\frac{11}{25}$$

$$\frac{1}{(s+1)^3(s+5)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{D}{s+5} = \frac{1}{64}\cdot\frac{1}{(s+1)} - \frac{1}{16}\cdot\frac{1}{(s+1)^2} + \frac{1}{4}\cdot\frac{1}{(s+1)^3} - \frac{1}{64}\cdot\frac{1}{(s+5)}$$

$$1 = A(s+1)^2(s+5) + B(s+1)(s+5) + C(s+5) + D(s+1)^3$$

if $s = -1$: $1 = 0 + 0 + C\cdot 4 + 0$ So $\boxed{C = \frac{1}{4}}$

if $s = -5$: $1 = 0 + 0 + 0 + D(-4)^3$ So $\boxed{D = -\frac{1}{64}}$

Comparing the coefficients in front of s^3: $0 = A + D$ So $\boxed{A = \frac{1}{64}}$

Comparing the constant term: $1 = 5A + 5B + 5C + D$ So $B = \frac{1}{5}\left(1 - \frac{5}{64} - \frac{5}{4} + \frac{1}{64}\right) =$

$$= \frac{1}{5}\left(1 - \frac{1}{16} - \frac{20}{16}\right) = \frac{1}{5}\left(\frac{16-21}{16}\right) = \boxed{-\frac{1}{16}}$$

1 cont.

So $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{11}{25}\mathcal{L}^{-1}\{\frac{1}{s}\} + \frac{1}{5}\mathcal{L}^{-1}\{\frac{1}{s^2}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{3}{50}\mathcal{L}^{-1}\{\frac{1}{s+5}\}$

$+ \mathcal{U}(t-1)\left(\frac{1}{64}\mathcal{L}^{-1}\{\frac{1}{s+1}\}(t-1) - \frac{1}{16}\mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}(t-1) + \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{(s+1)^3}\}(t-1) - \frac{1}{64}\mathcal{L}^{-1}\{\frac{1}{s+5}\}(t-1)\right) =$

$= e^{-t} - \frac{11}{25} + \frac{1}{5}t + \frac{1}{2}e^{-t} - \frac{3}{50}e^{-5t} + \mathcal{U}(t-1)\left[\frac{1}{64}e^{-(t-1)} - \frac{1}{16}e^{-t}(t-1) + \frac{1}{4}e^{-t}\cdot\frac{1}{2}(t-1)^2 - \frac{1}{64}e^{-5(t-1)}\right]$

$= \frac{3}{2}e^{-t} - \frac{11}{25} + \frac{1}{5}t - \frac{3}{50}e^{-5t} + \mathcal{U}(t-1)\left(\frac{1}{64}e^{-(t-1)} - \frac{1}{16}(t-1)e^{-t} + \frac{1}{8}(t-1)^2e^{-t} - \frac{1}{64}e^{-5(t-1)}\right)$

② $y'' + y' - 2y = t\cos t \qquad y(0) = 0\ , \quad y'(0) = 0$

$$s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + \left(sY(s) - \underbrace{y(0)}_{0}\right) - 2Y(s) = \mathcal{L}\{t\cdot\cos t\}$$

$$Y(s)\left(s^2 + s - 2\right) = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$Y(s)\,(s+2)(s-1) = -\frac{1\cdot(s^2+1) - s\cdot 2s}{(s^2+1)^2}$$

$$Y(s) = \frac{1}{(s+2)(s-1)} \cdot \frac{(s^2-1)}{(s^2+1)^2} = \frac{\cancel{(s-1)}(s+1)}{(s+2)\cancel{(s-1)}(s^2+1)^2}$$

$$\frac{s+1}{(s+2)(s^2+1)^2} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2}$$

$$(s+1) = A(s^2+1)^2 + (Bs+C)(s+2)(s^2+1) + (Ds+E)(s+2)$$

if $s = -2$: $\quad -1 = A(4+1)^2 + 0 + 0 \qquad$ So $\boxed{A = -\frac{1}{25}}$

s^4 : $\quad 0 = A + B \qquad$ So $\boxed{B = \frac{1}{25}}$

s^3 : $\quad 0 = C + 2B \qquad$ So $\boxed{C = -\frac{2}{25}}$

s^0 : $\quad 1 = A + 2C + 2E \qquad$ So $E = \frac{1}{2}(1 - A - 2C) =$

$= \frac{1}{2}\left(1 + \frac{1}{25} + \frac{4}{25}\right) = \frac{1}{2}\left(1 + \frac{\cancel{5}^{1}}{\cancel{25}_{5}}\right) =$

$= \frac{1}{2}\cdot\frac{6}{5} = \frac{3}{5} \qquad \boxed{E = \frac{3}{5}}$

s^1 : $\quad 1 = \underbrace{C + 2B}_{0} + E + 2D$

So $D = \frac{1}{2}(1 - E) = \frac{1}{2}\cdot\frac{2}{5} = \frac{1}{5}$

$\boxed{D = \frac{1}{5}}$

Thus $$\frac{s+1}{(s+2)(s^2+1)^2} = -\frac{1}{25}\cdot\frac{1}{(s+2)} + \frac{1}{25}\cdot\frac{s-2}{(s^2+1)} + \frac{1}{5}\cdot\frac{s+3}{(s^2+1)^2}$$