$\boxed{4.4}$ #4 $y'' + y' - 6y = 2x$

1) $t^2 + t - 6 = 0$

$(t-2)(t+3) = 0$

$t_1 = 2 \qquad t_2 = -3$

$y_{hom} = C_1 e^{2x} + C_2 e^{-3x}$

2) $y_p(x) = Ax + B$

$y_p' = A$

$y_p'' = 0$

$0 + A - 6(Ax + B) = 2x$

$A - 6Ax - 6B = 2x$

$\begin{cases} -6A = 2 \\ A - 6B = 0 \end{cases} \qquad A = -\frac{1}{3}$

$B = \frac{1}{6}A = -\frac{1}{18}$

$$\boxed{y(x) = C_1 e^{2x} + C_2 e^{-3x} - \frac{1}{3}x - \frac{1}{18}}$$

$\boxed{4.4}$ #10 $y'' + 2y' = 2x+5 - e^{-2x}$

1) $t^2 + 2t = 0$

$t(t+2) = 0$

$t_1 = 0 \qquad t_2 = -2$

$e^{0x} = 1 \qquad e^{-2x}$

$y_{hom} = c_1 + c_2 e^{-2x}$

2) $y'' + 2y' = 2x+5$

$y_{p_1} = Ax + B$ at first glance but since any constant is a solution of the homogeneous equation we need to look for

$y_{p_1} = x(Ax+B) = Ax^2 + Bx$

$y_{p_1}' = 2Ax + B$

$y_{p_1}'' = 2A$

$2A + 2(2Ax+B) = 2x+5$

$2A + 4Ax + 2B = 2x+5$

$\left| \begin{array}{l} 4A = 2 \\ 2A + 2B = 5 \end{array} \right. \qquad A = \frac{1}{2}$

$B = \frac{5-2A}{2} = \frac{5-1}{2} = \frac{4}{2} = 2$

$y_{p_1}(x) = \frac{1}{2}x^2 + 2x$

3) $y'' + 2y' = -e^{-2x}$

$y_{p_2} = Ae^{-2x}$ at first glance but since -2 is a zero of the algebraic equation of multiplicity 1 (i.e. e^{-2x} is a solution of the homogeneous equation) we need to look for

$y_{p_2} = Axe^{-2x} \qquad y_{p_2}' = Ae^{-2x} + Ax(-2)e^{-2x} = A(1-2x)e^{-2x}$

$\boxed{4.4}$ #10 cont.

$y_{p_2}'' = A(-2)e^{-2x} + A(1-2x)(-2)e^{-2x} = A(-2-2+4x)e^{-2x} = 4A(x-1)e^{-2x}$

$4A(x-1)e^{-2x} + 2A(1-2x)e^{-2x} = -e^{-2x}$ $\quad | \div e^{-2x}$

$\cancel{4Ax} - 4A + 2A - \cancel{4Ax} = -1$

$-2A = -1$

$A = \frac{1}{2}.$

$y_{p_2}(x) = \frac{1}{2}xe^{-2x}$

$$\boxed{y(x) = c_1 + c_2e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

$\boxed{4.4}$ #30 $\quad y'' + 4y' + 4y = (3+x)e^{-2x} \quad y(0) = 2,\ y'(0) = 5$

1) $t^2 + 4t + 4 = 0$

$(t+2)^2 = 0$

$t = -2$ with multiplicity 2

$y_{hom} = c_1e^{-2x} + c_2xe^{-2x}$

2) $y_p = (Ax+B)e^{-2x}$ at first glance but since -2 is a zero of multiplicity 2 of the algebraic equation (i.e., both e^{-2x} and xe^{-2x} are solutions of the homogeneous equation) we need to look for

$y_p = (Ax+B)x^2e^{-2x} = (Ax^3 + Bx^2)e^{-2x}$

$\boxed{4.4}$ #30 cont.

$$y_p' = (3Ax^2+2Bx)e^{-2x} - 2(Ax^3+Bx^2)e^{-2x} = (-2Ax^3+(3A-2B)x^2+2Bx)e^{-2x}$$

$$y_p'' = (-6Ax^2+(6A-4B)x+2B)e^{-2x} - 2(-2Ax^3+(3A-2B)x^2+2Bx)e^{-2x} =$$

$$= (4Ax^3+(-6A-6A+4B)x^2+(6A-4B-4B)x+2B)e^{-2x}$$

$$(4Ax^3+(-12A+4B)x^2+(6A-8B)x+2B)e^{-2x}$$

$$+4(-2Ax^3+(3A-2B)x^2+2Bx)e^{-2x} + 4(Ax^3+Bx^2)e^{-2x} = (3+x)e^{-2x} \Big/ \div e^{-2x}$$

$$\cancel{4Ax^3}+(-12A+4B)x^2+(6A-8B)x+2B-\cancel{8Ax^3}+(12A-8B)x^2+8Bx$$

$$+\cancel{4Ax^3}+4Bx^2 = x+3$$

$$(-\cancel{12A}+\cancel{4B}+\cancel{12A}-\cancel{8B}+\cancel{4B})x^2+(6A-\cancel{8B}+\cancel{8B})x+2B = x+3$$

$$6Ax+2B = x+3$$

$$\begin{cases} 6A = 1 \\ 2B = 3 \end{cases} \qquad A = \frac{1}{6} \qquad B = \frac{3}{2} \qquad y_p = \left(\frac{1}{6}x^3+\frac{3}{2}x^2\right)e^{-2x}$$

$$y(x) = c_1e^{-2x}+c_2xe^{-2x}+\left(\frac{1}{6}x^3+\frac{3}{2}x^2\right)e^{-2x} =$$

$$\boxed{y(x) = \left(\frac{1}{6}x^3+\frac{3}{2}x^2+c_2x+c_1\right)e^{-2x}}$$

$$y'(x) = \left(\frac{1}{2}x^2+3x+c_2\right)e^{-2x} - 2\left(\frac{1}{6}x^3+\frac{3}{2}x^2+c_2x+c_1\right)e^{-2x}$$

$$y(0) = 2 = c_1\cdot e^0 = c_1 \qquad \boxed{c_1 = 2}$$

$$y'(0) = 5 = c_2e^0 - 2\cdot c_1e^0 = c_2 - 2\cdot 2 = c_2 - 4$$

$$\boxed{c_2 = 9}$$

$$\boxed{y(x) = \left(\frac{1}{6}x^3+\frac{3}{2}x^2+9x+2\right)e^{-2x}}$$

Final ANSWER