

#2 4.2

$$y'' + 2y' + y = 0 \quad y_1 = xe^{-x}$$

$$y_2(x) = u(x) \cdot y_1(x) = u(x) \cdot xe^{-x}$$

$$y_2' = u' \cdot xe^{-x} + u(e^{-x} + x \cdot (-1)e^{-x}) = u'xe^{-x} + u(1-x)e^{-x}$$

$$y_2'' = u''xe^{-x} + u'(1-x)e^{-x} + u'(1-x)e^{-x} + u(-e^{-x} + (1-x)(-1)e^{-x})$$

$$= u''xe^{-x} + u'2(1-x)e^{-x} + u(-2+x)e^{-x}$$

$$u''xe^{-x} + u'2(1-x)e^{-x} + \cancel{u(x-2)e^{-x}} + 2u'xe^{-x} + \cancel{2u(1-x)e^{-x}} + \cancel{uxe^{-x}} = 0 \quad | : e^{-x}$$

$$u''x + 2(1-x)u' + 2xu' = 0$$

$$u''x + 2u' = 0$$

$$u''x = -2u' \quad v = u'$$

$$v'x = -2v$$

$$\frac{v'}{v} = -\frac{2}{x}$$

$$\int \frac{v'}{v} dx = -\int \frac{2}{x} dx + C$$

$$\ln|v| = -2 \ln|x| + C$$

$$\ln|v| = \ln\left(\frac{1}{x^2}\right) + C$$

$$|v| = \frac{1}{x^2} \cdot C_1$$

$$u' = \frac{C_1}{x^2}$$

$$\int u' dx = C_1 \int \frac{1}{x^2} dx + C_2$$

$$u = C_1 \int x^{-2} dx + C_2 = C_1 \frac{x^{-1}}{-1} + C_2 =$$

$$= C_3 \frac{1}{x} + C_2$$

$$\boxed{y_2(x) = e^{-x}}$$

pick $C_2 = 0$ $C_3 = 1$

$$u = \frac{1}{x} \quad y_2(x) = \frac{1}{x} \cdot xe^{-x} = e^{-x}$$

#12 4.2 $4x^2y'' + y = 0$ $y_1 = x^{\frac{1}{2}} \ln x$

$$y_2(x) = x^{\frac{1}{2}} \ln(x) \cdot u(x)$$

$$y_2' = \left[x^{\frac{1}{2}} \cdot \ln(x) \right]' \cdot u + x^{\frac{1}{2}} \ln x \cdot u' = \left(\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{\frac{1}{2}} \cdot \frac{1}{x} \right) u + x^{\frac{1}{2}} \ln x \cdot u' =$$

$$= \left(\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{-\frac{1}{2}} \right) u + x^{\frac{1}{2}} \ln x \cdot u'$$

$$y_2'' = \left(\frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-\frac{3}{2}} \ln x + \frac{1}{2} x^{-\frac{1}{2}} \cdot \frac{1}{x} + \left(+\frac{1}{2}\right) \cdot x^{-\frac{3}{2}} \right) u + \left(\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{-\frac{1}{2}} \right) u' +$$

$$+ \left(\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{\frac{1}{2}} \cdot \frac{1}{x} \right) u' + x^{\frac{1}{2}} \ln x \cdot u'' =$$

$$= \left(-\frac{1}{4} x^{-\frac{3}{2}} \ln x + \frac{1}{2} x^{-\frac{3}{2}} - \frac{1}{2} x^{-\frac{3}{2}} \right) u + 2 \left(\frac{1}{2} \ln x + 1 \right) x^{-\frac{1}{2}} u' + x^{\frac{1}{2}} \ln x u''$$

$$= -\frac{1}{4} x^{-\frac{3}{2}} \ln x u + (\ln x + 2) x^{-\frac{1}{2}} u' + x^{\frac{1}{2}} \ln x \cdot u''$$

$$4x^2 \left(-\frac{1}{4} x^{-\frac{3}{2}} \ln x u + (\ln(x) + 2) x^{-\frac{1}{2}} u' + x^{\frac{1}{2}} \ln x u'' \right) + x^{\frac{1}{2}} \ln x \cdot u = 0$$

$$-x^{\frac{1}{2}} \ln x u + 4x^{\frac{3}{2}} (\ln x + 2) u' + 4x^{\frac{5}{2}} \ln x u'' + x^{\frac{1}{2}} \ln x u = 0 \quad | : 4x^{\frac{3}{2}}$$

$$x \ln x u'' + (\ln x + 2) u' = 0 \quad v = u'$$

$$x \ln x v' + (\ln x + 2) v = 0$$

$$x \ln x v' = -(\ln x + 2) v$$

$$\frac{v'}{v} = -\frac{\ln x + 2}{x \ln x}$$

$$w = \ln x \quad dw = \frac{1}{x} dx$$

$$\int \frac{v'}{v} dx = - \int \frac{\ln x + 2}{\ln x} \cdot \left(\frac{1}{x} dx \right) + C$$

$y_2 = x^{\frac{1}{2}}$

$$\ln|v| = - \int \frac{w+2}{w} dw + C - \int \frac{w}{w} dw - \int \frac{2}{w} dw + C = -w - 2 \ln|w| + C$$

$$|v| = e^{-w} \cdot \frac{1}{w^2} \cdot C_1 = e^{-\ln x} \cdot \frac{1}{(\ln x)^2} \cdot C_1 = \frac{1}{x} \cdot \frac{1}{(\ln x)^2} \cdot C_1 \quad \text{pick } C_1 = -1, C_2 = 0$$

$$u' = \frac{C_1}{x(\ln x)^2} \quad u = C_1 \int \frac{dx}{x(\ln x)^2} + C_2 = C_1 \int \frac{1}{w^2} dw + C_2 = C_1 \cdot \frac{w^{-1}}{-1} + C_2 = -C_1 \frac{1}{\ln x} + C_2$$

4.3

#11 $y'' - 3y' + 2y = 0$

$$t^2 - 3t + 2 = 0$$

$$(t - 2)(t - 1) = 0$$

$$t_1 = 2 \quad t_2 = 1$$

$$y(x) = c_1 e^{2x} + c_2 e^x$$

4.3

#12 $2y'' + 2y' + y = 0$

$$2t^2 + 2t + 1 = 0$$

$$t_{1,2} = \frac{-2 \pm \sqrt{4 - 8}}{4} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm 2i}{4} = -\frac{1}{2} \pm \frac{1}{2}i$$

$$y(x) = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{1}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{1}{2}x\right)$$

4.3

#24 $y^{(4)} - 2y'' + y = 0$

$$t^4 - 2t^2 + 1 = 0$$

$$(t^2 - 1)^2 = 0$$

$$(t - 1)^2 (t + 1)^2 = 0$$

$t_1 = 1 \quad t_2 = -1$ each has multiplicity 2

$$\begin{matrix} e^x & e^{-x} \\ xe^x & xe^{-x} \end{matrix}$$

$$y(x) = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}$$

4.3

#34

$$y'' - 2y' + y = 0$$

$$y(0) = 5, \quad y'(0) = 10$$

$$t^2 - 2t + 1 = 0$$

$$(t-1)^2 = 0$$

$t = 1$ with multiplicity 2

$$y = c_1 e^x + c_2 x e^x$$

$$y' = c_1 e^x + c_2 e^x + c_2 x e^x$$

$$5 = c_1 e^0 + c_2 \cdot 0 e^0 = c_1$$

$$10 = c_1 e^0 + c_2 e^0 + c_2 \cdot 0 e^0 = c_1 + c_2$$

$$c_2 = 10 - c_1 = 10 - 5 = 5$$

$$y(x) = 5(1+x)e^x$$

4.3

#38

$$y'' + 4y = 0$$

$$y(0) = 0 \quad y(\pi) = 0$$

$$t^2 + 4 = 0$$

$$t^2 = -4$$

$$t_1 = 2i \quad t_2 = -2i$$

$$\alpha = 0 \quad \beta = 2$$

$$y(x) = c_1 e^{0x} \cos 2x + c_2 e^{0x} \sin 2x = c_1 \cos 2x + c_2 \sin 2x$$

$$0 = y(0) = c_1 \cdot \overset{1}{\cos 0} + c_2 \overset{0}{\sin 0} = c_1$$

$$0 = y(\pi) = c_1 \cdot \underset{1}{\cos 2\pi} + c_2 \underset{0}{\sin 2\pi} = c_1$$

So $c_1 = 0$ & c_2 could be any. The boundary value problem does not have unique solution. It has many solutions and

they are $y(x) = c \sin 2x$ where c can be any real constant