

#2 4.2

$$y'' + 2y' + y = 0 \quad y_1 = xe^{-x}$$

$$y_2(x) = u(x) \cdot y_1(x) = u(x) \cdot xe^{-x}$$

$$y_2' = u' \cdot xe^{-x} + u(e^{-x} + x \cdot (-1)e^{-x}) = u'xe^{-x} + u(1-x)e^{-x}$$

$$y_2'' = u''xe^{-x} + u'(1-x)e^{-x} + u'(1-x)e^{-x} + u(-e^{-x} + (1-x)(-1)e^{-x})$$

$$= u''xe^{-x} + u'2(1-x)e^{-x} + u(-2+x)e^{-x}$$

$$u''xe^{-x} + u'2(1-x)e^{-x} + \underline{u(x-2)e^{-x}} + 2u'xe^{-x} + \underline{2u(1-x)e^{-x}} + \underline{uxe^{-x}} = 0 \quad | : e^{-x}$$

$$u''x + 2(1-x)u' + 2xu' = 0$$

$$u''x + 2u' = 0$$

$$u''x = -2u' \quad v = u'$$

$$v'x = -2v$$

$$\frac{v'}{v} = -\frac{2}{x}$$

$$\int \frac{v'}{v} dx = - \int \frac{2}{x} dx + C$$

$$\ln|v| = -2 \ln|x| + C$$

$$\ln|v| = \ln\left(\frac{1}{x^2}\right) + C$$

$$|v| = \frac{1}{x^2} \cdot C_1$$

$$u' = \frac{C_1}{x^2}$$

$$\int u' dx = C_1 \int \frac{1}{x^2} dx + C_2$$

$$u = C_1 \int x^{-2} dx + C_2 = C_1 \frac{x^{-1}}{-1} + C_2 =$$

$$= C_3 \frac{1}{x} + C_2$$

$y_2(x) = e^{-x}$

pick $C_2 = 0$ $C_3 = 1$

$$u = \frac{1}{x} \quad y_2(x) = \frac{1}{x} \cdot xe^{-x} = e^{-x}$$

#10 4.2 $x^2 y'' + 2xy' - 6y = 0$ $y_1 = x^2$

$y_2(x) = \text{[scribble]} = x^2 \cdot u(x)$

$y_2' = \text{[scribble]} 2xu + x^2 u'$

$y_2'' = 2xu' + x^2 u'' + 2u + 2xu' = x^2 \cdot u'' + 4xu' + 2u$

$x^2(x^2 u'' + 4xu' + 2u) + 2x(x^2 u' + 2xu) - 6x^2 u = 0$

$x^4 \cdot u'' + 4x^3 u' + \underline{2x^2 u} + 2x^3 u' + \underline{4x^2 u} - \underline{6x^2 u} = 0$

$x^4 u'' + 6x^3 u' = 0 \quad | \div x^3$

$xu'' + 6u' = 0 \quad v = u'$

$xv' + 6v = 0$

$xv' = -6v$

$\frac{v'}{v} = -\frac{6}{x}$

$\int \frac{v'}{v} dx = -6 \int \frac{1}{x} dx + C$

$\ln|v| = -6 \ln|x| + C$

$|v| = \frac{1}{x^6} \cdot C_1$

$u' = \frac{C_1}{x^6}$ ~~particular~~

$\int u' dx = C_1 \int \frac{1}{x^6} dx + C_2$

$u = C_1 \int x^{-6} dx + C_2 = C_1 \frac{x^{-5}}{-5} + C_2 = -\frac{C_1}{5} \cdot \frac{1}{x^5} + C_2$

pick $C_1 = -5$ $C_2 = 0$ $u(x) = \frac{1}{x^5}$

$y_2(x) = x^2 \cdot \frac{1}{x^5} = \frac{1}{x^3}$

$y_2(x) = \frac{1}{x^3}$

#12 4.2 $4x^2y'' + y = 0$ $y_1 = x^{\frac{1}{2}} \ln x$

$$y_2(x) = x^{\frac{1}{2}} \ln(x) \cdot u(x)$$

$$y_2' = \left[x^{\frac{1}{2}} \cdot \ln(x) \right]' \cdot u + x^{\frac{1}{2}} \ln x \cdot u' = \left(\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{\frac{1}{2}} \cdot \frac{1}{x} \right) u + x^{\frac{1}{2}} \ln x \cdot u' =$$

$$= \left(\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{-\frac{1}{2}} \right) u + x^{\frac{1}{2}} \ln x \cdot u'$$

$$y_2'' = \left(\frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-\frac{3}{2}} \ln x + \frac{1}{2} x^{-\frac{1}{2}} \cdot \frac{1}{x} + \left(+\frac{1}{2}\right) \cdot x^{-\frac{3}{2}} \right) u + \left(\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{-\frac{1}{2}} \right) u' +$$

$$+ \left(\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{\frac{1}{2}} \cdot \frac{1}{x} \right) u' + x^{\frac{1}{2}} \ln x \cdot u'' =$$

$$= \left(-\frac{1}{4} x^{-\frac{3}{2}} \ln x + \frac{1}{2} x^{-\frac{3}{2}} - \frac{1}{2} x^{-\frac{3}{2}} \right) u + 2 \left(\frac{1}{2} \ln x + 1 \right) x^{-\frac{1}{2}} u' + x^{\frac{1}{2}} \ln x u''$$

$$= -\frac{1}{4} x^{-\frac{3}{2}} \ln x u + (\ln x + 2) x^{-\frac{1}{2}} u' + x^{\frac{1}{2}} \ln x \cdot u''$$

$$4x^2 \left(-\frac{1}{4} x^{-\frac{3}{2}} \ln x u + (\ln(x) + 2) x^{-\frac{1}{2}} u' + x^{\frac{1}{2}} \ln x u'' \right) + x^{\frac{1}{2}} \ln x \cdot u = 0$$

$$-x^{\frac{1}{2}} \ln x u + 4x^{\frac{3}{2}} (\ln x + 2) u' + 4x^{\frac{5}{2}} \ln x u'' + x^{\frac{1}{2}} \ln x u = 0 \quad | : 4x^{\frac{3}{2}}$$

$$x \ln x u'' + (\ln x + 2) u' = 0 \quad v = u'$$

$$x \ln x v' + (\ln x + 2) v = 0$$

$$x \ln x v' = -(\ln x + 2) v$$

$$\frac{v'}{v} = -\frac{\ln x + 2}{x \ln x}$$

$$w = \ln x \quad dw = \frac{1}{x} dx$$

$$\int \frac{v'}{v} dx = - \int \frac{\ln x + 2}{\ln x} \cdot \left(\frac{1}{x} dx \right) + C$$

$y_2 = x^{\frac{1}{2}}$

$$\ln|v| = - \int \frac{w+2}{w} dw + C - \int \frac{w}{w} dw - \int \frac{2}{w} dw + C = -w - 2 \ln|w| + C$$

$$|v| = e^{-w} \cdot \frac{1}{w^2} \cdot C_1 = e^{-\ln x} \cdot \frac{1}{(\ln x)^2} \cdot C_1 = \frac{1}{x} \cdot \frac{1}{(\ln x)^2} \cdot C_1 \quad \text{pick } C_1 = -1, C_2 = 0$$

$$u' = \frac{C_1}{x(\ln x)^2} \quad u = C_1 \int \frac{dx}{x(\ln x)^2} + C_2 = C_1 \int \frac{1}{w^2} dw + C_2 = C_1 \cdot \frac{w^{-1}}{-1} + C_2 = -C_1 \frac{1}{\ln x} + C_2$$