16.3 THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Recall the Fundamental theorem of Calculus:

If F is differentiable and F' is continuous, then

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

The analog for line integrals is:

If C is a smooth curve, $C: \vec{r} = \vec{r}(t)$, $a \le t \le b$, $f(x,y)$ is differentiable, and its gradient $\nabla f = (f'_x, f'_y)$ is continuous on C, then

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

proof:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \left(f'_x(x,y)\cdot\vec{i} + f'_y(x,y)\vec{j}\right)\cdot\left(x'(t)\vec{i} + y'(t)\cdot\vec{j}\right)dt =$$

$$= \int_a^b f'_x(x(t),y(t))\cdot x'(t) + f'_y(x(t),y(t))\cdot y'(t)\,dt = \int_a^b \left(f(x(t),y(t))\right)'_t\,dt =$$

$$= f(x(b),y(b)) - f(x(a),y(a)) = f(\vec{r}(b)) - f(\vec{r}(a))$$

where $\vec{r}(t) = (x(t), y(t))$

Fundamental theorem of Calculus for $F(t) := f(x(t),y(t))$

Analogous statement is true in 3D.

ex. 1 For the gravitational field $\vec{F}(\vec{r}) = -\frac{mMG}{|\vec{r}|^3} \cdot \vec{r}$ find the work done in moving particle with mass m from $(3,4,12)$ to $(2,2,0)$ along a smooth curve C.

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r}$$

But we know that gravitational field is conservative, i.e., it is of the form ∇f for the potential function

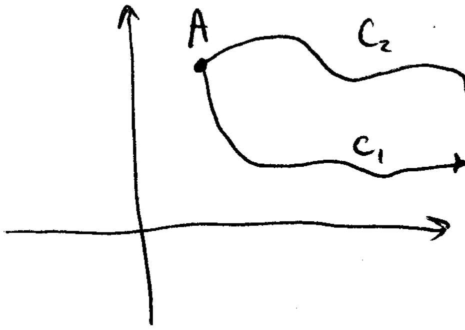
$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

Fund. theorem for line integrals

So $$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{\downarrow}{=} f(2,2,0) - f(3,4,12) =$$

$$= \frac{mMG}{\sqrt{4+4+0}} - \frac{mMG}{\sqrt{9+16+144}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)$$

Independence of path:



For an arbitrary vector field $\vec{F}$

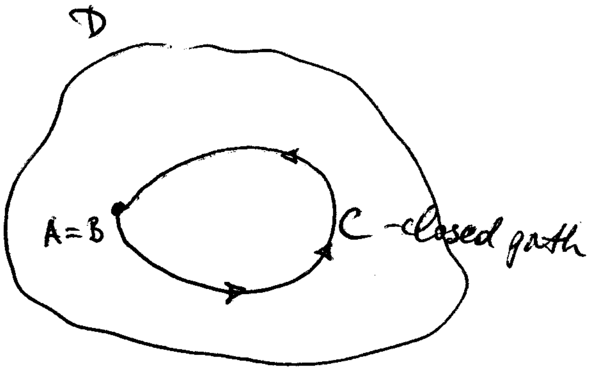
$$\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$$

But if $\vec{F}$ is conservative, i.e., $\vec{F} = \nabla f$ for some f - potential function, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = \int_{C_2} \nabla f \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

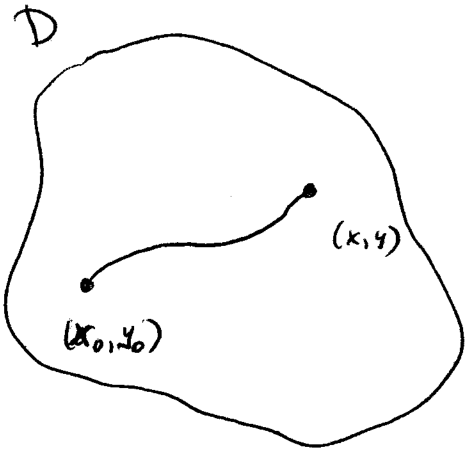
When $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two smooth curves C_1 and C_2 with common beginning and end we say that the integral $\int_C \vec{F} \cdot d\vec{r}$ is <u>independent of path</u>

<u>Fact (easy)</u>: $\int_C \vec{F} \cdot d\vec{r}$ is independent of path iff $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path.



curve is <u>closed</u> if the beginning and the end coincide

$\boxed{\text{Theorem}}$ If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path then $\vec{F}$ is a conservative vector field



Indeed, if we define $f(x,y) := \int_{C_{(x,y)}} \vec{F}\cdot d\vec{r}$

where $C_{(x,y)}$ is any smooth curve connecting a fixed point (x_0, y_0) and (x, y),

then $\nabla f = \vec{F}$

$\boxed{\text{Theorem}}$ If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\cdot\vec{j}$ is a conservative vector field, then

$$P'_y(x,y) = Q'_x(x,y) \qquad \text{(if those partials exist)}$$

proof: Since $\underset{\left(P(x,y),\, Q(x,y)\right)}{\vec{F}(x,y)} = \nabla f(x,y) = (f'_x, f'_y)$ for some potential $f(x,y)$

we have $P'_y = (f'_x)'_y = f''_{xy} = f''_{yx} = (f'_y)'_x = (Q)'_x$

$\boxed{\text{Theorem (reverse)}}$ If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\cdot\vec{j}$ and $P'_y(x,y) = Q'_x(x,y)$, then $\vec{F}(x,y)$ is conservative

$\boxed{ex.2}$ Is $\vec{F}(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$ conservative?

$\boxed{ex.3}$ (a) Is $\vec{F}(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$ conservative?

(b) If ~~it~~ it is, find potential $f(x,y)$ s.t. $\nabla f = \vec{F}$

(c) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $C: \vec{r}(t) = (e^t \sin t, e^t \cos t)$

$0 \le t \le \pi$