**Spring 2008, Math706 – Numerical Linear Algebra**

Homework 4

1. Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be nonsingular. Show that $\mathbf{A}$ has an LU factorization if and only if for each $k$ with $1 \leq k \leq m$, the upper-left $k \times k$ block $\mathbf{A}_{1:k,1:k}$ is nonsingular.

2. Suppose $\mathbf{A} \in \mathbb{C}^{m\times m}$ is banded with bandwidth $2p+1$, and a factorization $\mathbf{PA} = \mathbf{LU}$ is computed by Gaussian Elimination with partial pivoting. What can you say about the sparsity of $\mathbf{L}$ and $\mathbf{U}$?

3. Consider Gaussian Elimination carried out with pivoting by columns instead of rows, leading to a factorization $\mathbf{AQ} = \mathbf{LU}$ where $\mathbf{Q}$ is a permutation matrix

   (a) Show that if $\mathbf{A}$ is nonsingular, such a factorization always exists.

   (b) Give an example that $\mathbf{A}$ is singular and such a factorization does not exist.

4. Suppose $\mathbf{A} \in \mathbb{C}^{m\times m}$ is *strictly column diagonally dominant*, i.e.,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|.$$

   Show that if Gaussian Elimination with partial pivoting is applied to $\mathbf{A}$, no row interchange take place.

5. Show that for Gaussian Elimination with partial pivoting applied to any matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$, the growth factor satisfies $\rho \leq 2^{m-1}$.

6. Suppose $\mathbf{A} \in \mathbb{C}^{m\times m}$ is hermitian, or in the real case, symmetric.

   (a) Describe a strategy of *symmetric pivoting* to preserve the hermitian structure while still leading to a unit lower-triangular matrix with $|l_{ij}| \leq 1$.

   (b) What is the form of the matrix factorization computed by your algorithm?

   (c) What is its asymptotic operation count?

7. Using the proof of Theorem 16.2 (Page 118) as a guide, derive Theorem 23.3 (Page 177) from Theorems 23.2 (Page 176) and 17.1 (Page 122).