

Spring 2008, Math706 – Numerical Linear Algebra

Homework 1

- Let  $\mathbf{A} \in \mathbb{C}^{m \times m}$  be hermitian, i.e.,  $\mathbf{A}^* = \mathbf{A}$ .
  - Prove that all eigenvalues of  $\mathbf{A}$  are real.
  - Prove that if  $\mathbf{x}$  and  $\mathbf{y}$  are eigenvectors corresponding to distinct eigenvalues, then  $\mathbf{x}$  and  $\mathbf{y}$  are orthogonal.
- Let  $\mathbf{x} \in \mathbb{C}^m$  and  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , show that
  - $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{m}\|\mathbf{x}\|_\infty$ .
  - $\frac{1}{\sqrt{n}}\|\mathbf{A}\|_\infty \leq \|\mathbf{A}\|_2 \leq \sqrt{m}\|\mathbf{A}\|_\infty$ .
- Determine SVDs of the following matrices by hand calculation:

$$(a) \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$  with  $m \geq n$ , show that  $\mathbf{A}^*\mathbf{A}$  is nonsingular if and only if  $\mathbf{A}$  has full rank.
- \* Suppose  $\mathbf{A} \in \mathbb{C}^{m \times m}$  has an SVD  $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ . Find an eigenvalue decomposition of the  $2m \times 2m$  hermitian matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{A}^* \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$$

- Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- What is the orthogonal projector  $\mathbf{P}$  onto  $\text{range}(\mathbf{A})$ , and what is the image under  $\mathbf{P}$  of the vector  $(1, 2, 3)^*$ ?
  - Same questions for  $\mathbf{B}$ .
- Let  $\mathbf{P} \in \mathbb{C}^{m \times m}$  be a nonzero projector. Show that  $\|\mathbf{P}\|_2 \geq 1$ , with equality if and only if  $\mathbf{P}$  is an orthogonal projector.