Math 142 - Fall 2004 Solutions
Test#3
November 5, 2004

(15 points) Short Answer Questions. The following questions are based on the following definitions. All questions are independent of each other. Do not assume anything unless the question states it.

$$A = \sum_{n=1}^{\infty} a_n \quad A_n = a_1 + a_2 + \ldots + a_n \quad B = \sum_{n=1}^{\infty} b_n \quad B_n = b_1 + b_2 + \ldots + b_n$$

If the series A converges, then what do you know about the sequence $\{a_n\}$?

If the series A converges, the individual terms must converge to zero. We also have seen where $a_n \to 0$ does not mean that the sum exist (ie, harmonic series).

If the sequence of partial sums A_n converges to 17, then what do we know about the series A ?

This is the basic idea behind knowing what the sum of an infinite number of terms is equal to. We define the sum of an infinite number of terms to be equal to the limit of the partial sums. So if the partial sums are converging to 17, then the series is equal to 17.

If the sequence of partial sums B_n diverges, then what do we know about the series B ?

Similar to the above question, when the partial sums do not converge, the series is divergent.

When using the ratio test on the series A, what is the definition of ρ ?

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

When using the limit comparison test for the series A knowing B is a divergent series, what is the definition of L ?

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

When using the limit comparison test for the series A knowing B is a divergent series and $L = 0.25$, what does this tell us about A ?

In the limit comparison test, when $0 < L < \infty$ (and L is in this case) and we are comparing the series A to a series that we know is divergent, then A is divergent. If $L = 0$, then we could not have used the b_n to compare to a_n. We can still use the limit comparison test, but we have to select a better choice for our b_n.

How do we describe the situation when A is convergent, but $\sum |a_n|$ is divergent?

This type of convergence is called conditionally convergent. The ordering of the terms is critical. An example of this was the alternating series $\sum (-1)^{n+1}\frac{1}{n}$; it is convergent, but the harmonic series is not.

(15 points) Integral Test. Use the integral test to show the following series is convergent.

$$A = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The integral test is used to prove the partial sums are bounded. The partial sums can be estimated from above:

$$A_n = \sum_{k=1}^{n} a_k \le a_1 + \int_1^n \frac{1}{x^2} \quad dx = a_1 + 1 - \frac{1}{n}$$

This implies that the partial sums are bounded from above by $a_1 + 1$. The sequence of partial sums is also increasing, because the next term *in the sequence is just the previous term plus a positve number $\frac{1}{n}$. Theorem D tells us that an increasing sequence that is bounded has a limit. Therefore, A_n have a limit and this implies that the series A is convergent.*

We never proved this, but $A = \pi^2/6$. According to the integral test, if you use the first 30 terms of the series as an approximation, what is the estimate on the error ($A - A_{30}$) in using this approximation ?

We can use the integral test to also determine an estimate on the error if we use the first N terms of the series as an approximation. Notice that

$$\textit{Error } = A - A_{30} = a_{31} + a_{32} + a_{33} + \dots$$

Using the standard picture for this problem, the right side of the above equation are the rectangles that are under the graph of $f(x)$ starting at $x = 30$ and going off to the right. Again, using the fact that the integral can be thought of as area under the curve, an estimate of the error can be determined.

$$\textit{Error } \leq \int_{30}^{\infty} \frac{1}{x^2} \quad dx = \frac{1}{30}$$

So, the error in using the first 30 terms is less than $\frac{1}{30}$. For another series, the estimate of using an N-term approximation will not necessarily be $\frac{1}{N}$; it turned out that way for this problem.

(40 points) Tests, Tests, Tests. Use the given test to prove the convergence or divergence of the following integrals. The test may be inconclusive. Your answer should be a complete sentence and should include how you drew your conclusion.

$$\text{Limit Comparison: } A = \sum_{n=1}^{\infty} \frac{5n}{n^4 + 1}$$

Using $b_n = \frac{1}{n^3}$,

$$L = \lim_{n \to \infty} \frac{a_n}{b_n} = 5$$

First, $\sum b_n$ is convergent because of the p-test. Second, L is greater than zero and is a real number (mathematically, $0 < L < \infty$). These two facts imply that A is convergent.

$$\text{Limit Comparison: } B = \sum_{n=1}^{\infty} \sin(\frac{1}{n})$$

Using $b_n = \frac{1}{n}$,

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

This limit can be computed by using L'Hopital's Rule since you will have a zero-over-zero situation. First, $\sum b_n$ is the Harmonic series and is divergent. Second, L is greater than zero and is a real number (mathematically, $0 < L < \infty$). These two facts imply that A is divergent.
Note that if you using a divergent series and have $L = 0$, this test is inconclusive. The limit comparison test may still work, but you have to select a better choice for your b_n sequence.

$$\text{Ratio Test: } C = \sum_{n=1}^{\infty} \frac{10^n}{n!}$$

In this problem,

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{10}{n+1} = 0$$

In the ratio test, $\rho < 1$ implies that the series is convergent. In this problem, $\rho = 0$ so the series is convergent.

$$\text{Ratio Test: } D = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

In this problem,

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1$$

In the ratio test when $\rho = 1$, the test is inconclusive. This means that the test can not tell us what the series is doing. We know that this series is convergent from the p-test, but not from the ratio test.

(15 points) Power Series. Find the set of convergence for the following power series. Specify any test that you use to draw your conclusions.

$$A = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$$

To find the set of convergence, we first use the ratio test where $\rho < 1$ which provides a positive statement of when the series is convergent.

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}|x| = |x| < 1$$

Now, you check $x = -1$ and $x = 1$ (the endpoints of the above interval). These are the locations where $\rho = 1$, so the ratio test is inconclusive. This means we have to use another test to see if the series is convergent or divergent at these locations. For $x = -1$, $\sum(-1)^n \frac{1}{\sqrt{n}}$ can be shown to be convergent using the alternating series test. For $x = 1$, $\sum \frac{1}{\sqrt{n}}$ can be shown to be divergent using the p-test.
So, the set of convergence is $-1 \leq x < 1$.

(15 points) Taylor and Maclaurin Series.

Write out the first five terms of the Maclaurin series for $\sin(x)$.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$$

Write out the expression for the n-th term in this series.

Starting with $n = 1$, the n-th term in the above series is

$$(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$$

Notice that in the series for $\sin(x)$*, the terms are all odd (1,3,5, ...). The description of the n-th term must reflect that odd terms are used in this series.*

Write out the first five terms of the Maclaurin series for $\sin(\frac{x}{2})$.

$$\sin(x/2) = x - \frac{(x/2)^3}{3!} + \frac{(x/2)^5}{5!} - \frac{(x/2)^7}{7!} + \frac{(x/2)^9}{9!} + \ldots$$

Write out the first five terms of the Maclaurin series for $x\sin(x^2)$.

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} + \ldots$$

$$x\sin(x^2) = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \frac{x^{19}}{9!} + \ldots$$