Math 142 - Fall 2004 Solutions
Test#2
October 13, 2004

(15 points) Partial Fractions. Use partial fractions to decompose the following rational function, then evaluate the integral.

$$\frac{7s+4}{(s-2)(s+4)}$$

The first part of this problem is just asking for another way of representing the given rational function. If the denominator is factorable, then we can use partial fractions to break the function into a sum of simpler functions.

$$\frac{7s+4}{(s-2)(s+4)} = \frac{A}{s-2} + \frac{B}{s+4}$$
$$= \frac{A(s+4)+B(s-2)}{(s-2)(s+4)}$$
$$= \frac{(A+B)s+4A-2B}{(s-2)(s+4)}$$

Now we solve for A and B by setting up two equations using the coefficients of the polynomials that appear in the numerator.

$$A+B=7 \quad 4A-2B=4$$

By solving for A in the first equation and then plugging into the second, you can solve for B.

$$A=7-B \quad 4(7-B)-2B=4$$
$$28-4B-2B=4 \to 24=6B \to B=4 \quad A=3$$

$$\frac{7s+4}{(s-2)(s+4)} = \frac{3}{s-2} + \frac{4}{s+4}$$

Now using the above decomposition, we can do the second part of the problem.

$$\int \frac{7s+4}{(s-2)(s+4)} \quad ds = \int \frac{3}{s-2} + \frac{4}{s+4}$$
$$= 3\ln(s-2) + 4\ln(s+4) + C$$

(30 points) Integration by Parts. Evaluate the following integral.

$$\int x \cos 4x \quad dx$$

Remember that integration by parts requires you to determine the function u and v such that $\int u'v = uv - \int uv'$. In this problem,

$$\begin{array}{cc} u = x & v' = \cos 4x \quad dx \\ u' = dx & v = \frac{1}{4}\sin 4x \end{array}$$

$$\int x \cos 4x \quad dx = \frac{x}{4}\sin 4x - \int \frac{1}{4}\sin 4x \quad dx$$
$$= \frac{x}{4}\sin 4x + \frac{1}{16}\cos 4x + C$$

$$\int \tan^{-1} x \quad dx$$

Remember that integration by parts requires you to determine the function u and v such that $\int u'v = uv - \int uv'$. In this problem,

$$\begin{array}{cc} u = \tan^{-1} x & v' = dx \\ u' = \frac{dx}{1+x^2} & v = x \end{array}$$

$$\int \tan^{-1} x \quad dx = x\tan^{-1} x - \int \frac{x}{1+x^2} \quad dx$$
$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

(30 points) L'Hopital's Rule. Evaluate the following limits.

$$A = \lim_{t\to 0} \frac{t - \sin t}{\tan t}$$

This is a zero-over-zero case so we can use L'Hopital's Rule.

$$A = \lim_{t\to 0} \frac{1 - \cos t}{\sec^2 t} = \frac{0}{1}$$

Note that $\sec 0 = \frac{1}{\cos 0} = 1$.

$$B = \lim_{y \to 2} \frac{y^2 + 6}{y - 2}$$

Note that the numerator approaches 10 as $y \to 2$ while the denominator approaches 0. This is not a case that we use L'Hopital's Rule. This limit is undefined.

$$C = \lim_{z \to 1} \frac{z^2 + 4z - 5}{z^3 - 1}$$

This is a zero-over-zero case. From algebra, you should remember that a polynomial $P(x)$ that has a root at $x = a$ has a factor of $(x - a)$. We could factor each term, notice that there is a factor of $(x - 1)$ that cancels out and determine the limit. But we have L'Hopital's Rule to make this task simpler.

$$C = \lim_{z \to 1} \frac{2z + 4}{3z^2} = \frac{6}{3} = 2$$

$$D = \lim_{t \to \infty} \frac{\ln(t^2 + 5t)}{\ln t}$$

L'Hopital's Rule works for infinity-over-infinity cases as well.

$$D = \lim_{t \to \infty} \frac{\frac{2t+5}{t^2+5t}}{\frac{1}{t}} = \lim_{t \to \infty} \frac{2t^2 + 5t}{t^2 + 5t} = \lim_{t \to \infty} \frac{4t + 5}{2t} = \lim_{t \to \infty} \frac{4}{2} = 2$$

$$E = \lim_{r \to 0} (1 + 3r)^{\frac{1}{r}}$$

This is not immediately a L'Hopitals case. But with a little modification using the natural logarithm function, we have

$$\ln E = \lim_{r \to 0} \frac{1}{r} \ln(1 + 3r) = \lim_{r \to 0} \frac{\ln(1 + 3r)}{r}$$

This is a case that we can apply L'Hopital's Rule.

$$\ln E = \lim_{r \to 0} \frac{\frac{3}{1+3r}}{1} = 3$$
$$E = e^3$$

(15 points) Sequences. Write out the first four terms of the sequence and determine the limit of the sequence. Assume all these sequences converge and have a limit.

$$a_n = \frac{2n+1}{n+10}$$

$$a_0 = \frac{1}{10} \quad a_1 = \frac{3}{11} \quad a_2 = \frac{5}{12} \quad a_3 = \frac{7}{13}$$
$$\lim_{n \to \infty} \frac{2n+1}{n+10} = \lim_{n \to \infty} \frac{2 + \frac{1}{n}}{1 + \frac{10}{n}} = 2$$

$$b_n = (\frac{1}{3}) \cdot b_{n-1} \text{ and } b_1 = 90$$

$$b_1 = 90 \quad b_2 = \frac{1}{3} \cdot 90 = 30 \quad b_3 = 10 \quad b_4 = \frac{10}{3}$$

$lim_{n \to \infty} b_n = 0$ since we could rewrite the definition of $b_n = 90 \frac{1}{3^{n-1}}$. As $n \to \infty$, the denominator is growing larger and larger, so we have the sequence of points approaching zero.

$$c_n = \sqrt{12 + 4c_{n-1}} \text{ and } c_1 = 2$$

Since we know that the sequence of c_n have a limit, then we know that once the sequence gets close to its limit, the recursive formula should produce a value that is close to the limit. If we let C be the limit of the sequence, then C will satisfy the following equation.

$$C = \sqrt{12 + 4C}$$
$$C^2 = 12 + 4C$$

$$C^2 - 4C - 12 = 0$$

$$(C-6)(C+2) = 0 \rightarrow C = 6 \ or \ C = -2$$

Notice that each term in the sequence is the positive square root of a number, hence the limit is 6, not -2.

(10 points) Sequences and Series.

Let E be a series and E_n be the partial sums of E. Explain why the limit of E_n as $n \rightarrow \infty$ converges. This will show that E converges. Hint: Write out the 8th term of each of the sequence and use it to generalize your explanation.

$$E = \sum \frac{1}{n!} \qquad E_n = 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}$$

Compare the partial sums of this sequence to the geometric series $C_n = \sum \frac{1}{2^k}$. *You can show the the partial sums* E_n *are each less than* C_n *which is bounded by 2.* E_n *is also an increasing sequence. From Theorem D, an increasing sequence that is bounded from above has a limit. If the sequence of partial sums has a limit, then the series converges.*

Let H *be the harmonic series and* H_n *be the partial sums of* H. *Explain why the limit of* H_n *as* $n \rightarrow \infty$ *diverges. This will show that* H *diverges. Hint: Write out the 8th term of each of the sequence and use it to generalize your explanation.*

$$H = \sum \frac{1}{n} \qquad H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$$

Notice that for every integer, there exist a smallest power of two that is greater than or equal to the given integer. For example, if given 5, then 8 is the next power of two. Using this fact, we create a sequence of partial sums D_n *that is less than the sequence of* H_n *term by term. Here is* D_6 *in comparison to* $H - 6$, *for example.*

$$H_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$D_6 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

Grouping the terms within D_8 *according to the denominator, you see that we have groups that sum to* $\frac{1}{2}$ *and continuing this process, we see that the sequence of* D_n *are unbounded since we are adding one-half over and over again. So, the sequence* D_n *is unbounded, hence the sequence* H_n *is unbounded. In conclusion, if* H_n *have no limit, then the series* H *(harmonic series) is divergent.*