Answer Key
Test #1
September 18, 2003

Please print your name and the date at the top of each page. Show all work. Your grade will be based on you showing that you know the proper methods and steps in solving the problems.

(15 points) Evaluate each of the limits. Write out your steps.

$$\lim_{q \to 3} \frac{2q^2 - 7q + 3}{q^2 + 4q - 21}$$

Since evaluating this rational function gives you a "zero-over-zero" case, we factor the numerator and the denominator, canceled out the common term and re-evaluate.

$$\lim_{q \to 3} \frac{(q-3)(2q-1)}{(q-3)(q+7)} = \lim_{q \to 3} \frac{(2q-1)}{(q+7)} = \frac{5}{10} = \frac{1}{2}$$

$$\lim_{\theta \to 0} \sin(4\theta)\csc(3\theta)$$

First, we change the $\csc(3\theta)$ into an expression using $\sin(3\theta)$. Then, we will take an advantage of the $\lim_{x \to 0} \frac{\sin x}{x} = 0$ theorem. We need to introduce 4θ and 3θ to the expression to let us use the theorem.

$$\lim_{\theta \to 0} \sin(4\theta)\csc(3\theta) = \lim_{\theta \to 0} \frac{\sin(4\theta)}{\sin(3\theta)} = \lim_{\theta \to 0} \frac{\sin(4\theta)}{\sin(3\theta)} \frac{4\theta}{4\theta} \frac{3\theta}{3\theta} =$$

$$\lim_{\theta \to 0} \frac{\sin(4\theta)}{4\theta} \frac{3\theta}{\sin(3\theta)} \frac{4\theta}{3\theta} = \lim_{\theta \to 0} \frac{\sin(4\theta)}{4\theta} \frac{3\theta}{\sin(3\theta)} \frac{4}{3} = 1 \cdot 1 \cdot \frac{4}{3} = \frac{4}{3}$$

$$\lim_{t\to\infty} \frac{t^2-3t+1}{4t^2-7t+8}$$

Here, we have a case where as t is getting larger both the numerator and denominator are getting very large. But by performing the right algebraic manipulations, you get to evaluate the limit.

$$\lim_{t\to\infty} \frac{t^2-3t+1}{4t^2-7t+8} = \lim_{t\to\infty} \frac{t^2-3t+1}{4t^2-7t+8}\left(\frac{\frac{1}{t^2}}{\frac{1}{t^2}}\right) = \lim_{t\to\infty} \frac{1-3\frac{1}{t}+\frac{1}{t^2}}{4-7\frac{1}{t}+8\frac{1}{t^2}} = \frac{1}{4}$$

(10 points) Using the definition, determine the derivative of $f(x)$. Write out your steps.

$$f(x) = \frac{1}{x}$$

First, write the definition of the deriviative, then perform algreba until you can evaluate the limit.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \left(\tfrac{1}{x+h} - \tfrac{1}{x}\right)/h \\ &= \lim_{h\to 0} \left(\tfrac{x-(x+h)}{(x+h)(x)}\right)/h \\ &= \lim_{h\to 0} \tfrac{-h}{(x+h)(x)(h)} \\ &= \lim_{h\to 0} \tfrac{-1}{(x+h)(x)} \\ &= \tfrac{-1}{x^2} \end{aligned}$$

(30 points) Using the rules and theorems, determine $\frac{dy}{dx}$. Show all work.

$$y = 4x^8 - 12x^3 - 20x + 17$$

Using the power rules for derivatives.

$$y' = 32x^7 - 36x^2 - 20$$

First, change the square root symbol into a power of 1/2. Then use the power rule for taking the derivative.

$$y = 8\sqrt{x} = 8x^{1/2}$$

$$y' = 8 \cdot \frac{1}{2}x^{-1/2} = \frac{4}{\sqrt{x}}$$

Use the product rule for taking the derivative.

$$y = (x^2 - 4x + 5)(x^2 + 1)$$

$$\begin{aligned} y' &= (x^2 - 4x + 5)'(x^2 + 1) + (x^2 - 4x + 5)(x^2 + 1)' \\ &= (2x - 4)(x^2 + 1) + (x^2 - 4x + 5)(2x) \\ &= (2x^3 - 4x^2 + 2x - 4) + (2x^3 - 8x^2 + 10x) \\ &= 4x^3 - 12x^2 + 12x - 4 \end{aligned}$$

Here we use the division rule to take the derivative of a rational function.

$$y = \frac{x^3}{x^2 - 3x + 1}$$

$$\begin{aligned} y' &= \tfrac{(x^3)'(x^2-3x+1)-(x^3)(x^2-3x+1)'}{(x^2-3x+1)^2} \\ &= \tfrac{(3x^2)(x^2-3x+1)-(x^3)(2x-3)}{(x^2-3x+1)^2} \\ &= \tfrac{(3x^4-9x^3+3x^2)-(2x^4-3x^3)}{(x^2-3x+1)^2} \\ &= \tfrac{x^4-6x^3+3x^2}{(x^2-3x+1)^2} \end{aligned}$$

These are the basic trigonometric functions.

$$y = 4\sin(x) + 3\cos(x) + 6\tan(x)$$

$$y' = 4\cos(x) - 3\sin(x) + 6\sec^2(x)$$

(15 points). Using the chain rule and other derivative rules, determine $\frac{dy}{dx}$. Show all work.

$$y = (x^3 + 4x^2 - 3x + 17)^4$$

$$y' = 4(x^3 + 4x^2 - 3x + 17)^3(3x^2 + 8x - 3)$$

$$y = \sqrt{x^2 + 4} = (x^2 + 4)^{1/2}$$

$$y' = \frac{1}{2}(x^2 + 4)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 4}}$$

$$y = \sin(4x + 3)$$

$$y' = \cos(4x + 3) \cdot 4 = 4\cos(4x + 3)$$

(10 points) Determine the line ("$y = mx + b$" form) that is tangent to the following function at $x = 4$.

$$f(x) = x^2 + 5x + 4$$

We need two pieces of information to determine the tangent line. First, the point where the line will be just touching (tangent) to the parabola, then the slope of the line. The slope of the tangent line at $x = 4$ will be the same as the first derivative, $f'(x)$, at $x = 4$ or $f'(4)$.

$$f(4) = 16 + 20 + 4 = 40$$

So, the point that the tangent line and the parabola have in common is $(4, 40)$.

$$f'(x) = 2x + 5$$

So, the derivative at $x = 4$ is $f'(4) = 2 \cdot 4 + 5 = 13$. Now, with the point $(4, 40)$ and $m = 13$, we use the point-slope formula to determine the line with slope 13 going through the point $(4, 40)$.

$$\begin{gathered} y - y_0 = m(x - x_0) \\ y - 40 = 13(x - 4) \\ y - 40 = 13x - 52 \\ y = 13x - 12 \end{gathered}$$

(20 points) Using implicit differentation, determine $\frac{dy}{dx}$. Show all steps.

$$4x^3 + 5y^3 = 10$$

$$\begin{gathered} 4x^3 + 5y^3 = 10 \\ 12x^2 + 15y^2 \cdot \frac{dy}{dx} = 0 \\ 15y^2 \cdot \frac{dy}{dx} = -12x^2 \\ \frac{dy}{dx} = \frac{-12x^2}{15y^2} = \frac{-4x^2}{5y^2} \end{gathered}$$

$$x^2y + xy^2 = 17$$

$$\begin{gathered} x^2y + xy^2 = 17 \\ 2xy + x^2\frac{dy}{dx} + y^2 + x \cdot 2y\frac{dy}{dx} = 0 \\ x^2\frac{dy}{dx} + x \cdot 2y\frac{dy}{dx} = -(2xy + y^2) \\ (x^2 + 2xy)\frac{dy}{dx} = -(2xy + y^2) \\ \frac{dy}{dx} = -\frac{2xy+y^2}{x^2+2xy} \end{gathered}$$