Worksheet #9: Rates of Change

Spring 2007

Objectives

- Know the definition of instantaneous rate of change
- Examine the different between average rates and instantaneous rates
- Learn how to estimate the instantaneous rate of change
- Learn the geometric meaning of the instantaneous rate of change
- Learn how to interpret the instantaneous rate in an application

Work in groups, but answer in your own words.

- Using $[xMin, xMax] = [0, 10]$, graph:

$$y = f(t) = (t - 8)(t - 5)(t - 3)t + 100$$

- Look in your book for the definition/formula of average rate of change. What is the formula for average rate of change of $f(t)$ between $t = 1$ and $t = 5$? Compute this average rate of change with the $f(t)$ above. In words, what is the meaning of the average rate of change?

$$\textit{Average Rate of Change} = \frac{f(5) - f(1)}{5 - 1} = \frac{56}{4} = 14$$

It is the slope of the line segment connecting the two points that are on the graph at $t = 1$ and $t = 5$. It is the rate (speed) of how fast the function is changing over time. It is not how much it changed, but the rate of the change.

- Graphically, what does this number represent? Hint: where have you seen this type of formula before?

Again, it is the slope of a line segment.

- What is your defintion of "instantaneous"?

Something is instantaneuous when it happens in an instant. Not over a length of time, but at an exact moment.

Remember that we are discussing rates of change. Your speedometer measures your car's rate of change. It is the change in position (in miles) over time (in hours). On a trip to Atlanta, it might take you 3 hours to drive 210 miles. Your average speed is 70 miles per hour. Your speedometer is measuring your instantaneous rate of change. It tells you how fast you are going right when you are looking at it. During the trip, it changes and reports 65 mph (when you first get on the highway), 35 mph (when you are traffic around Augusta), and 75 mph (when you are trying to make up time).

Use this example to understand the difference between average rate of change and instantaneous rate of change.

- If the variable t is in minutes, what is the time span used in the average above? Would you call this instantaneous?

The time span in the above problem is 4 minutes. In many applications, I would not think that's not very instantaneous. 4 minutes in a basketball game is not instantaneous.

- Give a time span that is nearly instantaneous starting at $t = 3$. Compute the average rate of change over this time span.

Instantaneous would be something happening at $t = 3$*. No, time span - it is happening at that exact moment. But if something started at* $t = 3$ *and finished at* $t = 3.1$*, that might be nearly instantaneous. But you might think, that 0.1 minutes is 10% of a minute, which is 6 seconds. In some applications, 6 seconds is a long time. So, perhaps, the time span from* $t = 3$ *to* $t = 3.0008$ *is a better choice of being nearly instantaneous.*

With a time span, you can compute the average rate of change.

$$\textit{Average Rate of Change} = \frac{f(3.008) - f(3)}{3.008 - 3}$$

$$= \frac{0.2393}{.008} = 29.91$$

- Give a time spand that is nearly instantaneous starting at $t = 6$. Compute the average rates of change over this time span using the function from above.

Maybe from $t = 6$ *to* $t = 6.002$ *is nearly instantaneous. With a time span, you can compute the average rate of change.*

$$\textit{Average Rate of Change} = \frac{f(6.002) - f(6)}{6.002 - 6}$$

$$= \frac{-0.07197}{.002} = -35.98$$

- Since the time span is nearly instantaneous, we can say that the two averages above approximations to the instantaneous rate of change at $t = 3$ and at $t = 6$.

Since the time span is nearly instantaneous, the average rate over this small amount of time is approximately the instantaneous rate of change. This is how we will estimate the instantaneous rate of change.

It is like measuring your average speed over a 10 second period by dividing the distance that you travelled by 10 seconds (then converting to mph). That average is probably close to your instantaneous rate of change at any point during that time span.

- Write a definition of the instantaneous rate of change of a function at $t = a$. Write a formula that approximates the instantaneous rate of change of a function $f(t)$ at $t = a$. In words, what is the meaning of the instantaneous rate of change?

The instantaneous rate of change in a function is the measurement of how fast the function is changing at a specific point.

$$\textit{Instantaneous Rate} = \frac{f(a+h) - f(a)}{h} \quad \textit{where } h \textit{ is small}$$

It's meaning is the same as the average rate of change, but the time span is very small, so you understand what is happening at an exact moment in time.

- Calculator feature. You can draw a line that is tangent to your graph at any point by using the "2nd-Draw-Tangent" feature. Just type in the x-value. Draw the tangent lines that go through the graph at $(3, 100)$ and $(6, 64)$.

These tangent lines essentially replicate the function at a very, very fine scale. If you zoom in on the function at $t = 3$, then you will notice that the function, eventhough it is not a line, behaves like a line at a very small scale. The tangent line mimicks the behavior on the function at a microscopic scale. The slope of this tangent line can be estimated by using $t = 3$ and $t = 3.0007$.

- Calculator feature. You can compute the slope of these lines by using the "2nd-Calc-6" feature on your calculator. Just type in the x-value. Compute the slopes of the two tangent lines above.

- Compare your answers from when you estimated the instantaneous rate of change to the slopes of the two tangent lines. The answers should be nearly the same. If they are, what might be wrong? What is another definition of the instantaneous rate of change?

The slopes of the tangent lines should be approximately the instantaneous rates of change that you found earlier. If they are close, then your interval of what you called nearly instantaneous needs to be shortened.

Slope at $t = 3 \; = 30$ *Slope at* $t = 3 \; = -36$

- As a class, we will have different feelings of what nearly instantaneous means. Compare your time span that you used for nearly instantaneous with your classmates. Eventhough this might have been different, were your answers to the other numerical questions much different than theirs? They shouldn't have been, so think about why they weren't different.