Worksheet #26 - Integrals and Expected Value

Spring 2007

Objectives

- Review exponential distributions
- Review expectations
- Learning how to use the integral to determine expectations

Background - Expectations Suppose you are a contestant on a game show. You are selected to roll a dice (six sided), but on each side of the die, there is a dollar amount. You will receive the amount that appears face up after your roll. The sides have the following amounts on them: \$100, \$200, \$200, \$400, \$400, \$800. The typical question would be to ask how much are you expected to win or how much on average does the show has to give away when a contestant plays this game.

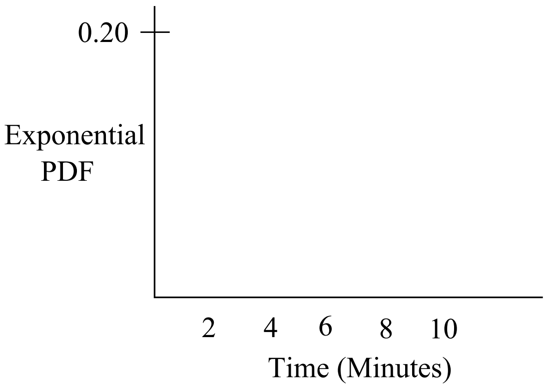
- Fill out the following chart. The expectation for each value is the value times the probability that that event will happen.

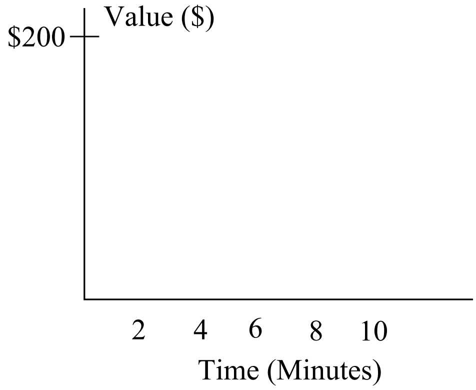
Event	Face Value	Probability	Expectation
ω_1	\$100		
ω_2	\$200		
ω_3	\$400		
ω_4	\$800		
		Total	

- Total the expectations. This is the expected value of playing the game. Think of this as a weighted average.

$$\text{Expected Value } = \sum V(\omega_i) \cdot P(\omega_i)$$

- Notice that the expected value is a sum. In this case it is just the sum of 4 numbers, but if the events were not discrete (finite number of terms), then we will have to subdivide the events into small finite number of groups and approximate the expected value. What mathematical tool will we probably use? Look at the title of this worksheet.





Background - Exponential Distributions Recall the typical problem that we have studied in regards to exponential distributions. Suppose you own a business and are tracking the number of customers that come through your door; you know from previous experience that you usually see about 12 customers per hour. Answer the following questions.

- What is the average wait time between customers? What units did you use? Convert this to minutes.

- What is the probability density function given the average wait time from the previous answer? Graph this function.

- What is the probability that you will have to wait more than 3 minutes but less than 4 minutes on the next customer? Express the probabilty being asked as an integral. What is the function being integrated? What are the integrands (the lower and upper limits of the integration)? Remember that your units have to agree.

- Convert the wait time into seconds. Re-write the probability density function.

- Write the integral that is equivalent to the probability that you will have to wait between 3 and 4 minutes. Remember your units have to match.

Expected Value Suppose there is a cost or value placed on amount of time that it takes to see the next customer. The value might be increasing or decreasing depending on the application. Suppose that the value of the customer is given by the following function:

$$V(t) = 200 - 20t \text{ for } t \leq 10$$

where t is in minutes. For t greater than 10 minutes, the value of the value function is zero. Graph this function.

The goal is to compute the expected value of a customer. Again, it will be the sum of the value at a certain time (like the value of the roll of a dice) times the probability. Fill out the following chart and notice that it is similar to the one that you filled out above.

Event	Approximate Value	Probability	Expectation
$\omega_1 = \{0 \le t \le 2\}$	$V(1) =$		
$\omega_2 = \{2 \le t \le 4\}$	$V(3) =$		
$\omega_3 = \{4 \le t \le 6\}$	$V(5) =$		
$\omega_4 = \{6 \le t \le 8\}$	$V(7) =$		
$\omega_5 = \{8 \le t \le 10\}$	$V(9) =$		
$\omega_6 = \{t \ge 10\}$	$V(t) =$		

Using the Integral with Expected Value As we have done in other applications, we need to subdivide the possible events into small cases. In the problem above, we will divide the time axis between $t = 0$ and $t = 10$ into a very large number of very small time intervals.

$$\begin{aligned}\text{Expected Value } &\approx \sum V(t_i) \cdot P(t_{i-1} \le W \le t_i) \\ &\approx \sum V(t_i) \cdot p(t_i) \cdot dt \\ &\approx \int V(t) \cdot p(t_i) dt\end{aligned}$$

In our problem, the expected value can be computed by:

$$\begin{aligned}\text{Expected Value } &= \int V(t) \cdot p(t) dt = \int_0^{10} (200 - 20t) \cdot \frac{1}{5} e^{-t/5} dt \\ &= \$113.53\end{aligned}$$

Homework: Use the following value function and average wait times (λ) to compute the expected average. Assume the value function is zero for $t \ge 10$.

$$V(t) = 2(t - 10)^2 \quad \lambda = 5$$

$$V(t) = 200 - 20t \quad \lambda = 2$$

$$V(t) = 200 - 20t \quad \lambda = 1$$