Worksheet #16: Derivatives

Spring 2007

Objectives

- Learn about the derivative of the product of two functions.
- Learn about the derivative of the division of two functions.

Recall the product rule is:

$$P = f(x) \cdot g(x) \qquad\qquad P' = f'(x)g(x) + f(x)g'(x)$$

And the division rule is:

$$Q = \frac{f(x)}{g(x)} \qquad\qquad Q' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

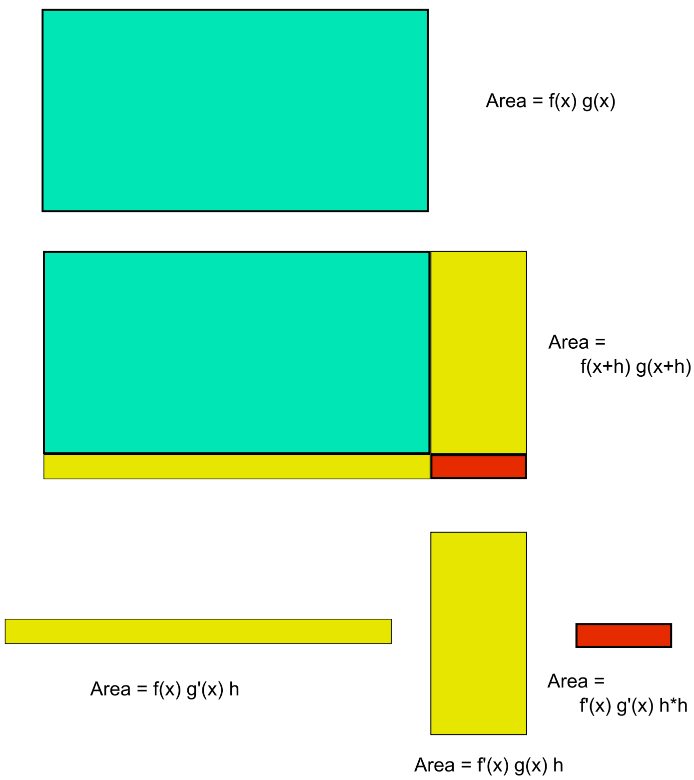

Figure 1: Use these pictures to help prove the product rule.

Proof of Product Rule. Here is a proof of why the product rule is true.

- Label the width of the first rectangle in figure $f(x)$ and the height $g(x)$. What is the area of the rectangle?

The area of the rectangle is width times length. So,

$$A = f(x) \cdot g(x)$$

- Approximate $f(x+h)$ and $g(x+h)$ using the derivative of $f(x)$ and $g(x)$. Label the lengths of the second rectangle with these approximations.

From class notes, we know:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

So, from this we can get:

$$f(x+h) \approx f(x) + h \cdot f'(x)$$
$$g(x+h) \approx g(x) + h \cdot g'(x)$$

This means that the extra width and length are approximately $h \cdot f'(x)$ *and* $h \cdot g'(x)$.

- Approximate the area of the second rectangle.

Again, the area of a rectangle is just width times length. So, for the second rectangle.

$$W = f(x) + h \cdot f'(x)$$

$$L = g(x) + h \cdot g'(x)$$

$$A = W \cdot L = f(x)g(x) + h \cdot f'(x) \cdot g(x) + h \cdot f(x) \cdot g'(x) + h^2 f'(x)g'(x)$$

- The derivative of the $f(x)g(x)$ can be approximated using the slope formula that we've used previously to approximate derivatives.

$$\frac{d}{dx}\left[f(x)g(x)\right] \approx \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

- The right side of this expression is the difference of the areas of the two rectangles divided by h. The right side of this expression can be expressed as the sum of three smaller rectangles divided by h. The right side can be simplified in such a way that the h in the denominator no longer exist. Simplify the right side.

$$\frac{d}{dx}\left[f(x)g(x)\right] \approx \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$\approx \frac{h \cdot f'(x) \cdot g(x) + h \cdot f(x) \cdot g'(x) + h^2 f'(x)g'(x)}{h}$$
$$\approx f'(x) \cdot g(x) + f(x) \cdot g'(x) + hf'(x)g'(x)$$

- When h approaches zero, what does the right side equal?

So, when h gets really smaller (nearly zero), the last term in the line above becomes practically zero. And we have proven,

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Problems. Determine the derivative. Write out the components of the product or quotient. Take the derivative of each component. Then write out the derivative.

$y = 40xe^{-0.5x}$ What is y' at $x = 0$? What is y' at $x = 4$?

$$y' = 40e^{-0.5x} - 20xe^{-0.5x} = 20e^{-0.5x}\left[2 - x\right]$$
$$y'(0) = 20\cdot 1\cdot 2 = 40$$
$$y'(4) = 20\cdot e^{-2}\cdot -2 = -5.413$$

This means y is increasing at $x = 0$ and decreasing at $x = 4$. It turns out that y takes its maximum value at $x = 2$ when $y' = 0$ - we should find the second derivative and verify that it is negative at $x = 2$ to justify that the critical point $x = 2$ is a local maximum.

$w = 20t^2e^{-0.25t}$ What is w' at $t = 1$ When is $w' = 0$?

$$w' = 40t\cdot e^{-0.25t} - 5t^2\cdot e^{-0.25t} = 5t\cdot e^{-0.25t}\left[8 - t\right]$$
$$w'(1) = 5\cdot e^{-0.25}\cdot 9 = 27.258$$
$$w'(t) = 0 \textit{ when either } t = 0 \textit{ or } t = 8.$$

$z = \frac{x^2-2x+5}{x^2+1}$ What is z' at $x = 10$? When is $z' = 0$?

$$z' = \frac{2x^2 - 8x - 2}{(x^2+1)^2}$$
$$z'(10) = \frac{200 - 80 - 2}{101^1} = \frac{118}{10201}$$

$z'(x) = 0$ when $2x^2 - 8x - 2 = 0$ and so we should either factor, complete the square or use quadratic formula. Let's complete the square.

$$2x^2 - 8x - 2 = 0$$
$$x^2 - 4x - 1 = 0$$
$$x^2 - 4x + 4 = 1 + 4$$
$$(x-2)^2 = 5$$
$$x = 2 \pm \sqrt{5}$$

$q = \frac{5t^2+2000}{t}$ Is q increasing at $t = 5$ When is $q' = 0$?

$$q' = \frac{5t^2 - 2000}{t^2} = 5 - \frac{2000}{t^2}$$

$$q'(5) = 5 - 80 = -75$$

$q'(t) = 0$ when $5 - \frac{2000}{t^2} = 0$. So, solve:

$$5 - \frac{2000}{t^2} = 0$$

$$5 = \frac{2000}{t^2}$$

$$t^2 = 400$$

$$t = \pm 20$$

This problem doesn't require using the quotient rule if you noticed that

$$q = \frac{5t^2 + 2000}{t} = 5t + \frac{2000}{t}$$

which requires only the power rule. But you should get the same formula for $q'(t)$.