A Reverse Isoperimetric Inequality, Stability and Extremal Theorems for Plane Curves with Bounded Curvature

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In this note we discuss some elementary theorems about the relation between area and length of closed embedded plane curves with bounded curvature. Our main result (see Theorem 4.1) solves the extremal problem of which domain has *largest* boundary length among embedded disks in the plane whose boundary curvatures are uniformly bounded and whose area is fixed and sufficiently small.

Reverse Isoperimetric Inequality. *If M is an embedded closed disk in the plane $\mathbf{R}^2$ whose boundary curvature satisfies $|\kappa| \le 1$ and with area $A \le \pi + 2\sqrt{3}$ then the length of $\dot{M}$ is bounded by*

$$\frac{L - 2\pi}{4} \le \operatorname{Arcsin}\left(\frac{A - \pi}{4}\right).$$

If equality holds then M is congruent to the peanut shaped domain of Figure 1.

This gives an estimate in the reverse direction to the classical isoperimetric inequality. There is also a threshold phenomenon: if the area is larger than $\pi + 2\sqrt{3}$ then then is no upper bound for the length of $\dot{M}$. This is the area the pinched peanut domain $P_{\sqrt{3}}$. Examples can be found by breaking a thin peanut and connecting the ends with a long narrow strip. In fact, the set of possible points (A, L) for embedded disks whose boundary satisfies $|\kappa| \le 1$ is further restricted (Theorem 4.1).

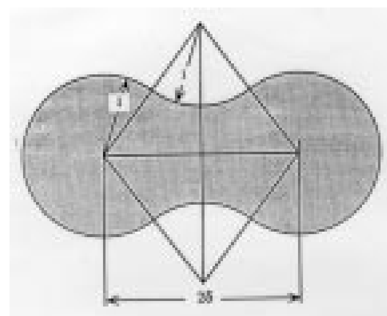

Fig. 1 "Peanut" domain P_δ.

In Section 4 we prove existence and uniqueness of the extremal figures. We use a replacement argument to show that extremals are piecewice circular arcs. Compactness depends on apriori length bounds. In Section 3 we consider length estimates and some related stability results in the class of embedded disks whose boundary curvature is uniformly bounded. Our results say that if area or some other quantity is small, such as the circumradius, then the curve must be near the circle. The results depend on a theorem of Pestov and Ionin on the existence of a large disk in a domain with uniformly bounded curvature. In Section 2 we include an argument for Pestov and Ionin's theorem along the lines of Lagunov's proof of the higher dimensional generalization using analysis of the structure of the cut set of such a domain. We indicate how the argument carries over to general Riemannian surfaces. Our results use both the existence of a disk and structure of the cut set. In Section 1 we consider curves which are only continuously differentiable and whose curvature is bounded in an appropriate weak sense which is suitable to extremal problems. Some other extremal problems for such curves have been studied previously. For example, the problem of finding the shortest plane curve with given endpoint and starting line element (position and direction) considered by Markov. The shortest plane curve given starting and ending line elements was found by Dubins.

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