CONVEX BODIES OF CONSTANT WIDTH AND CONSTANT BRIGHTNESS

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A *convex body* in $\mathbf{R}^n$ is a compact convex set with non-empty interior. A convex body K in the three dimensional Euclidean space has *constant width* w iff the orthogonal projection of K onto every line is an interval of length w. It has has *constant brightness* b iff the orthogonal projection of K onto every plane is a region of area b.

Theorem 1. *Any convex body in $\mathbf{R}^3$ of constant width and constant brightness is a Euclidean ball.*

Under the extra assumption that the boundary ∂K is C^2 this was proven by S. Nakajima [4] in 1926. Since then the problem of determining if there is a non-smooth non-spherical convex body in $\mathbf{R}^3$ of constant width and constant brightness has become well known among geometers studying convexity. For example see [2, Ques. 2 p. 437], [3].

In the case of bodies with C^2 boundaries and positive curvature Nakajima's result was generalized by Chakerian [1] in 1967 to "relative geometry" where the width and brightness are measured with with respect to some convex body K_0 symmetric about the origin called the *gauge body*. The following isolates the properties of the gauge body needed for the proof of Theorem 1 to generalize. Recall the *Minkowski sum* of two subsets A and B of $\mathbf{R}^n$ is $A + B = \{a + b : a \in A, b \in B\}$.

Definition. A convex body K_0 is a *regular gauge* iff it is centrally symmetric about the origin and there are convex sets K_1, K_2 and Euclidean balls B_r and B_R such that $K_0 = K_1 + B_r$ and $B_R = K_0 + K_2$.

Any convex K_0 body symmetric about the origin with C^2 boundary and positive Gaussian curvature is a regular gauge. For any linear subspace P of $\mathbf{R}^n$ let $K|P$ be the projection of K onto P (all projections in this paper are orthogonal). For any unit vector u let $w_K(u)$ be the width in the direction of u. For each positive integer k and any Borel subset of $\mathbf{R}^n$ be $V_k(A)$ be the k-dimensional volume of A (which in this paper is defined to be the k dimensional Hausdorff measure of A). Two subsets A and B are *homothic* iff there is a positive scalar λ and a vector v_0 so that $B = v_0 + \lambda A$.

Theorem 2. *Let K_0 be a regular gauge in $\mathbf{R}^3$ and let K be any convex body in $\mathbf{R}^3$ such that for some constants α, β*

$$w_K(u) = \alpha w_{K_0}(u), \quad V_2(K|u^\perp) = \beta V_2(K|u^\perp)$$

for all $u \in \mathbb{S}^{n-1}$. Then K is homothic to K_0.

References

1. G. D. Chakerian, *Sets of constant relative width and constant relative brightness*, Trans. Amer. Math. Soc. **129** (1967), 26–37. MR 35 #3545
2. R. J. Gardner, *Geometric tomography*, Notices Amer. Math. Soc. **42** (1995), no. 4, 422–429. MR **97b:**52003
3. ———, *Geometric tomography*, Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 1995. MR **96j:**52006
4. S. Nakajima, *Eine charakteristicische Eigenschaft der Kugel*, Jber. Deutsche Math.-Verein **35** (1926), 298–300.

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