

EXTREMAL APPROXIMATELY CONVEX FUNCTIONS AND ESTIMATING THE SIZE OF CONVEX HULLS

S. J. DILWORTH, RALPH HOWARD, AND JAMES W. ROBERTS

ABSTRACT. A real valued function f defined on a convex K is an *ap-proximately convex function* iff it satisfies

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + 1.$$

A though study of approximately convex functions is made. The principal results are a sharp universal upper bound for lower semi-continuous approximately convex functions that vanish on the vertices of a simplex and an explicit description of the unique largest bounded approximately convex function E vanishing on the vertices of a simplex.

A set A in a normed space is an *approximately convex set* iff for all $a, b \in A$ the distance of the midpoint $(a+b)/2$ to A is ≤ 1 . The bounds on approximately convex functions are used to show that in \mathbf{R}^n with the Euclidean norm, for any approximately convex set A , any point z of the convex hull of A is at a distance of at most $\lceil \log_2(n-1) \rceil + 1 + (n-1)/2^{\lceil \log_2(n-1) \rceil}$ is from A . Examples are given to show this is the sharp bound. Bounds for general norms on \mathbf{R}^n are also given.

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