Analysis on Homogeneous Spaces
Class Notes Spring 1994
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Contents

CHAPTER 1

Introduction

These are notes based very loosely on a class I gave on harmonic analysis on homogeneous spaces at the Royal Institute of Technology in Stockholm in the spring of 1994 as part of a course whose first term was a course in integral geometry (which is why there are so many references to integral geometry). They are not quite elementary in that it is assumed that the reader knows the basics of elementary differential geometry, that is the definition of a smooth manifold, vector fields and their flows, integration of differential forms, and some very elementary facts of Riemannian geometry. Volume one of Spivak [**25**] covers much much more than is required. As time, energy, and interest permits I plan to add to these notes mostly along the lines of applications to concrete problems on concrete spaces such as Spheres and Grassmann manifolds.

The main goal was to give a proof of the basic facts of harmonic analysis on compact symmetric spaces as given by the results in Chapter 5 (and Theorems 5.1.1 and 5.2.1 in particular) and then to apply these to concrete problems involving things such as the Radon and related transforms on these spaces. In this the notes are only half successful in that I an quite happy with the proofs in Chapter 5 in that they only use basic functional analysis and avoid the machinery of Lie groups and should be accessible to anyone with a year of graduate real analysis under their belt (and willing to take a few facts about manifolds on faith). As to the applications these notes are more or less a failure as none of any substance are given. Much of the class was spent on these applications, but as I just more or less followed standard presentations (mostly the wonderful book of Helgason [**17**]) there seemed little reason for writing up those lectures.

To make up for the lack of applications to Radon transforms on symmetric spaces, Appendix B uses the machinery of Chapter 5 in the case of finite groups and gives several results on Radon transforms on Grassmannians of subspaces of vector spaces over finite fields. I had a great deal of fun working this out and would like to think it is at least least moderately entertaining to read.

The two appendices C and D give a proof of Federer's coarea formula and use it to prove some Sobolev and Poincare type inequalities due to Federer and Fleming, Cheeger, Mckean, and Yau. These appendices (which were originally notes from an integral geometry class) are included in the belief that having “analysis” in the title obligates me to include some nontrivial inequalities.

As to the basic notation if M is a smooth manifold (and all manifolds are assume to be Hausdorff and paracompact) then the tangent bundle of M will be denoted by $T(M)$ and the tangent space at $x \in M$ by $T(M)_x$. If $f : M \to N$ is a smooth map then the derivative map is denoted by f_* so that f_{*x} is a linear map $f_{*x} : T(M)_x \to T(N)_{f(x)}$. The Lie bracket of two vector fields X and Y on M is denoted by $[X, Y]$.

CHAPTER 2

Basics about Lie Groups and Homogeneous Spaces

2.1. Definitions, Invariant Vector Fields and Forms

A ***Lie Group*** is a smooth manifold G and a smooth map $(\xi, \eta) \mapsto \xi\eta$ (the product) that makes G into a group. That is there is an element $e \in G$ so that $e\xi = \xi e = \xi$ for all $\xi \in G$. For any $\xi \in G$ there is an inverse ξ^{-1} so that $\xi\xi^{-1} = \xi^{-1}\xi = e$ and the associative law $\xi(\eta\zeta) = (\xi\eta)\zeta$ holds.

REMARK 2.1.1. According to Helgason [**16**, p. 153] the global definition of a Lie group given just given was emphasized until the 1920's when the basic properties where developed by H. Weyl, É. Cartan, and O. Schrier. Local versions of Lie groups have been around at least since the work of Lie in the nineteenth century.

EXERCISE 2.1.2. Use the implicit function theorem to show that the map $\xi \mapsto \xi^{-1}$ is smooth. HINT: This is easier if you know the formula for the derivative of the product map $(\xi, \eta) \mapsto \xi\eta$ given in proposition 2.1.5 below (whose proof does not use that $\xi \mapsto \xi^{-1}$ is smooth). $\square$

The ***left translation*** by $g \in G$ is the map $L_g(\xi) = g\xi$. This is smooth and has $L_{g^{-1}}$ as an inverse so it is a diffeomorphism of G with its self. Likewise there is ***right translation*** $R_g(\xi) = \xi g$. These satisfy

$$L_{g_1 g_2} = L_{g_1} L_{g_2}, \quad R_{g_1 g_2} = R_{g_2} R_{g_1}.$$

(Note that order of the products is reversed by right translation.) Also left and right translation commute

$$R_{g_1} \circ L_{g_2} = L_{g_2} \circ R_{g_1}.$$

A vector field is ***left invariant*** iff $(L_{g*}X)(\xi) = X(g\xi)$ for all $g, \xi \in G$. Denote by $\mathfrak{g}$ the vector space of all left invariant vector fields.

PROPOSITION 2.1.3. *If $v \in T(G)_e$ there is a unique left invariant vector field X with $X(e) = v$. Thus the dimension of $\mathfrak{g}$ as a vector space is* $\dim G$. *If c is a curve in G with $c(0) = e$ and $c'(0) = v$ then X is given by*

$$X(\xi) = \frac{d}{dt}\xi c(t)\bigg|_{t=0} = \frac{d}{dt}R_{c(t)}\xi\bigg|_{t=0} = L_{\xi*}v.$$

PROOF. Uniqueness is clear from the left invariance: If two left invariant vector fields agree at a point they are equal. To show existence just define $X(\xi) = L_{\xi*}v$ and verify that it is left invariant. To show the other formula for the left invariant extension holds define a vector field by $Y(\xi) := (d/dt)\xi c(t)|_{t=0}$. Then

$$L_{g*}Y(\xi) = L_{g*}\frac{d}{dt}\xi c(t)\bigg|_{t=0} = \frac{d}{dt}L_g\xi c(t)\bigg|_{t=0} = \frac{d}{dt}g\xi c(t)\bigg|_{t=0} = Y(g\xi).$$

So Y is left invariant and as $Y(e) = v = X(e)$ this implies $X = Y$ and completes the proof. $\square$

PROPOSITION 2.1.4. *Any left invariant vector field is complete (i. e. integral curves are defined of all of* $\mathbf{R}$.*) If X is left invariant and c in an integral curve of X with $c(0) = e$, then $c(s+t) = c(s)c(t)$. (That is c is a* ***one parameter subgroup*** *of G.)*

PROOF. Let $c : (a, b) \to G$ be an integral curve of the left invariant vector field X. We need to show that the domain of c can be extended to all of $\mathbf{R}$. Let $a < t_0 < t_1 < b$ and let $g \in G$ be the element so that $gc(t_0) = c(t_1)$. Define $\gamma : (a + (t_1 - t_0), b + (t_1 - t_0)) \to G$ be $\gamma(t) = gc(t - (t_1 - t_0))$. Then

$$\begin{aligned}\gamma'(t) &= L_{g*}c'(t - (t_1 - t_0)) = L_{g*}X(c(t - (t_1 - t_0)))\\ &= X(gc(t - (t_1 - t_0))) = X(\gamma(t))\end{aligned}$$

so γ is also an integral curve for X and as $\gamma(t_1) = gc(t_0) = c(t_1)$ this implies that $c = \gamma$ on the intersection of their domains. Letting $\delta = (t_1 - t_0,$ thus shows that c can be extended to $(a, b + \delta)$ by letting $c = \gamma$ on $[b, b + \delta)$. Repeating this argument k times shows that c can be extended as an integral curve of X to $(a, b + k\delta)$. Letting $k \to \infty$ shows that c can be extended to (a, ∞). A similar argument now shows that c can be extended to $\mathbf{R} = (-\infty, \infty)$. This completes the proof X is complete.

Let $s \in \mathbf{R}$ and let c be a integral curve of the left invariant vector field X with $c(0) = e$. Define $\gamma(t) = c(s)^{-1}c(s + t)$. Then $\gamma(0) = c(s)^{-1}c(s) = e$ and

$$\begin{aligned}\gamma'(t) &= L_{c(s)^{-1}*}c'(s + t) = L_{c(s)^{-1}*}X(c(s + t)\\ &= X(c(s)^{-1}c(s + t)) = X(\gamma(t)).\end{aligned}$$

Therefore by the uniqueness of integral curves for a vector field $\gamma(t) = c(s)^{-1}c(s + t) = c(t)$, which implies $c(s + t) = c(s)c(t)$. $\square$

If $v \in T(G)_e$ and X is the left invariant vector field extending v, then the one parameter subgroup c determined by X is usually denoted by $\exp(tv) := c(t)$. With this notation the map $v \mapsto c(1) = \exp(v)$ from $T(G)_e$ to G is the ***exponential map***.

PROPOSITION 2.1.5. *Let $p : G \times G \to G$ be the product map $p(\xi, \eta) = \xi\eta$. Then the derivative of p is given by*

$$p_{(\xi,\eta)*}(X, Y) = L_{\xi*}Y + R_{\eta*}X.$$

(Here the tangent space to $T(G\times G)_{(\xi,\eta)}$ is identified with $T(G)_\xi \times T(G)_\eta$ in the obvious way.) If $\iota : G \to G$ is the inverse map $\iota(\xi)=\xi^{-1}$, then

$$\iota_{\xi*}X = -L_{\xi^{-1}*}R_{\xi^{-1}}X = -R_{\xi^{-1}*}L_{\xi^{-1}}X.$$

Proof. To prove the formula for p_* it is enough to show that

$$p_{(\xi,\eta)*}(X,0) = R_{\eta*}X \quad \text{and} \quad p_{(\xi,\eta)*}(0,Y) = L_{\xi*}Y.$$

Let $c(t)$ be a curve in G with $c(0)=\xi$ and $c'(0)=X$. Then

$$f_{(\xi,\eta)*}(X,0) = \frac{d}{dt}p(c(t),\eta)\bigg|_{t=0} = \frac{d}{dt}c(t)\eta\bigg|_{t=0} = \frac{d}{dt}R_\eta c(t)\bigg|_{t=0} = R_{\eta*}X.$$

The calculation for $p_{(\xi,\eta)*}(0,Y) = L_{\xi*}Y$ is similar.

By Exercise 2.1.2 the map ι is smooth. To find the derivative of ι let $c(t)$ be a curve with $c(0)=\xi$ and $c'(0)=X$. Then $c(t)^{-1}c(t)\equiv e$ so

$$\begin{aligned}
0 &= \frac{d}{dt}(c(t)^{-1}c*(t)\bigg|_{t=0} \\
&= L_{c(0)^{-1}*}c'(0) + R_{c(0)*}\,\frac{d}{dt}c(t)^{-1}\bigg|_{t=0} \\
&= L_{\xi^{-1}*}X + R_{\xi*}\,\frac{d}{dt}c(t)^{-1}\bigg|_{t=0}.
\end{aligned}$$

Solving this for $(d/dt)c(t)^{-1}|_{t=0}$

$$\iota_{\xi*}X = \frac{d}{dt}c(t)^{-1}\bigg|_{t=0} = -R_{\xi^{-1}*}L_{\xi^{-1}*}X$$

□

Proposition 2.1.6. *If X and Y are left invariant vector fields, then so is $[X,Y]$.*

Proof. For any diffeomorphism φ and any vector fields the relation $\varphi_*[X,Y]=[\varphi_*X,\varphi_*Y]$ holds. If X and Y are left invariant the proposition follows by letting $\varphi = L_g$. □

Remark 2.1.7. The last proposition shows that the vector space $\mathfrak{g}$ of left invariant vector fields is closed under the Lie bracket. A ***Lie algebra*** vector space with a bilinear product $[\cdot,\dots]$ which is skew-symmetric (i.e. $[X,Y]=-[Y,X]$ that satisfies the ***Jacobi identity***

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]].$$

As the Lie bracket of vectors fields satisfies the Jacobi identity the last proposition show that $\mathfrak{g}$ is a Lie algebra, called ***the Lie algebra of*** G. While this is very important in some parts of the theory of Lie groups, for example the representation of Lie groups, it does not play much of a role in classically integral geometry.

Just as there are left and right invariant vector fields, there are left and right invariant forms. Let ω be a differential form on G Then ω is ***left invariant*** iff $L_g^*\omega = \omega$ for all $g \in G$. It is ***right invariant*** iff $R_g^*\omega = \omega$ for all $g \in G$.

PROPOSITION 2.1.8. *Let ω_0 be an element of $\bigwedge^k T^*(G)_e$. Then there is a unique left invariant form ω with $\omega_e = \omega_0$. If ω a left invariant form, then so is $R_g^*\omega$ for any $g \in G$.*

PROOF. Define ω by $\omega_\xi = L_{\xi^{-1}}^*\omega_0$. Then ω is easily checked to be left invariant.

Assume that ω is left invariant so that $L_\xi^*\omega = \omega$. Then using that left and right translation commute we have $L_x i^* R_g^*\omega = R_g^* L_\xi^*\omega = R_g^*\omega$. Therefore $R_g^*\omega$ is left invariant. □

2.2. Invariant Volume Forms and the Modular Function

Proposition 2.1.8 implies the space of left invariant volume forms (that is forms of degree n where $n = \dim G$) is one dimensional and that if Ω_G is a left invariant volume form, then so is $R_g^*\Omega_G$ for any $g \in G$. This allows us to define a function $\Delta_G^+ : G \to \mathbf{R}^\#$ (where $\mathbf{R}^\# = \mathbf{R} \setminus \{0\}$ is the ***multiplicative group of non-zero real numbers***) by

$$\Delta_G^+(g)\Omega_G = R_{g^{-1}}^*\Omega_G$$

where Ω_G is any non-zero left invariant volume form on G. It is easily checked this definition is independent of the choice of the form Ω_G. Also define the ***modular function*** Δ_G of G by

$$\Delta_G(g) := |\Delta_G^+(g)|.$$

The group G is ***unimodular*** iff $\Delta_G \equiv 1$. We will see shortly that G is unimodular iff there is measure on G that is both left and right invariant.

PROPOSITION 2.2.1. *The function Δ_G^+ never vanishes and is a smooth group homomorphism of G into $\mathbf{R}^\#$ (i.e. $\Delta_G^+(g_1g_2) = \Delta_G^+(g_1)\Delta_G^+(g_2)$). The function Δ_G is a smooth group homomorphism from G into $\mathbf{R}^+$ (the **multiplicative group of positive real numbers**). If G is connected then Δ_G^+ is positive on G. If K is a compact subgroup of G, then $\Delta_G^+(a) = \pm 1$ for all $a \in K$. In particular if G is compact then G is unimodular.*

PROOF. If $g_1, g_2 \in G$, then

$$\begin{aligned}\Delta_G^+(g_1g_2)\Omega_G &= R_{(g_1g_2)^{-1}}^*\Omega_G \\ &= R_{g_1^{-1}}^* R_{g_2^{-1}}^*\Omega_G \\ &= \Delta_G^+(g_1)\Delta_G^+(g_2)\Omega_G.\end{aligned}$$

This implies $\Delta_G^+(g_1g_2) = \Delta_G^+(g_1)\Delta_G^+(g_2)$. As $\Delta_G^+(e) = 1$ the relation $1 = \Delta_G^+(g)\Delta_G^+(g^{-1})$ implies $\Delta_G^+(g) \neq 0$. Thus Δ_G^+ is a homomorphism into $\mathbf{R}^\#$ as claimed. This implies Δ_G is a homomorphism into $\mathbf{R}^+$. If G is connected, then Δ_G^+ can not change sign with out taking on the value zero. Therefore in this case $\Delta_G^+ > 0$. If K is a compact subgroup of G, then the image $\Delta_G^+[K]$ is a compact subgroup of $\mathbf{R}^\#$. But every compact subgroup of $\mathbf{R}^\#$ is a subset of $\{\pm 1\}$. This completes proof. $\square$

PROPOSITION 2.2.2. *Let $\Omega_G \neq 0$ be a left invariant volume form on G, and Θ a right invariant volume form with $\Theta_e = \Omega_G$. Then*

$$\Theta = \Delta_G^+ \Omega_G.$$

Therefore G has a volume form that is invariant under both left and right translations if and only if $\Delta_G^+ \equiv 1$.

PROOF. Note

$$\left(R_g^* \Delta_G^+ \Omega_G\right)_\xi = \Delta_G^+(\xi g^{-1}) R_g^* \Omega_G = \Delta_G^+(\xi g^{-1}) \Delta_G^+(g) \Omega_G = \Delta_G^+(\xi)(\Omega_G)_\xi.$$

Thus $\Delta_G^+ \Omega_G$ is right invariant. As this form and Θ both equal Ω_G at the origin right invariance implies $\Delta_G^+ \Omega_G = \Theta$. $\square$

REMARK 2.2.3. In many cases (see examples below) it is straight forward to find left and right invariant volume forms on the group G. Then the last proposition gives an easy method for finding the function Δ_G^+.

PROPOSITION 2.2.4. *If Ω_G is a left invariant volume form on G, and $\iota : G \to G$ is the map $\iota(\xi) = \xi^{-1}$, then*

$$\iota^* \Omega_G = (-1)^n \Delta_G^+ \Omega_G.$$

PROOF. First note that $\iota \circ R_g = L_{g^{-1}} \circ \iota$. Thus

$$R_g^* \iota^* \Omega_G = (\iota \circ R_g)^* \Omega_G = (L_{g^{-1}} \circ \iota)^* \Omega_G = \iota^* L_{g^{-1}}^* \Omega_G = \iota^* \Omega_G$$

which shows that $\iota^* \Omega_G$ is right invariant. The derivative of ι at e is $\iota_{e*} = -\operatorname{Id}$ (cf. Prop. 2.1.5) and thus $(\iota^* \Omega_G)_e = (-1)^n (\Omega_G)_e$. The result now follows from the last proposition. $\square$

Let $d\xi$ be the left invariant measure on G, which can be viewed as the "absolute value" of a left invariant volume form Ω_G. Then the transformation rules above can be summarized as

$$\boxed{\begin{aligned} \int_G f(g\xi)\, d\xi &= \int_G f(\xi)\, d\xi \\ \int_G f(\xi g)\, d\xi &= \Delta_G(g) \int_G f(\xi)\, d\xi \\ \int_G f(\xi^{-1})\, d\xi &= \int_G f(\xi) \Delta_G(\xi)\, d\xi \end{aligned}} \tag{2.1}$$

2.3. Homogeneous Spaces

2.3.1. Definitions and the closed subgroup theorem. Here we describe the spaces that have a transitive action by a Lie group G. All these spaces can be realized as spaces of cosets $G/K := \{\xi K : \xi \in G\}$ for closed subgroups K of G. The the closed subgroups of a Lie group are better behaved that one might expect at first because of:

THEOREM 2.3.1 (É. Cartan). *A closed subgroup H of a Lie group G is a **Lie subgroup** of G. That is H is am imbedded submanifold of G in such the manifold topology of H is the same as the subspace topology.*

REMARK 2.3.2. This result was first proven by É Cartan. A little earlier Von Neumann had proven the result in the case $G = GL(n, \mathbf{R})$. The proof here follows Sternberg [**27**, p. 228] and is based on several lemmas. As most of the closed subgroups of Lie groups that we will encounter will more or less obviously be Lie subgroups the reader will lose little in skipping the the proof. And to be honest we will be using two facts ((2.2) and (2.3)) which are standard parts of the basics about Lie groups, but which get not proof here.

LEMMA 2.3.3. *Let $\mathfrak{g}$ be the Lie algebra of G. Let $\{X_l\}$ be a sequence of elements of $\mathfrak{g}$ so that $\lim_{l\to\infty} X_l = X$ for some $X \in \mathfrak{g}$ and assume there is a sequence of nonzero real numbers t_l with $\lim_{l\to\infty} t_l = 0$ and so that $\exp(t_l X_l) \in H$ for all l. Then $\exp(tX) \in H$ for all t.*

PROOF. As $\exp(-X_l) = (\exp(X_l))^{-1}$ by possibly replacing X_l by $-X_l$ we can assume $t_l > 0$. Letting $[\cdot]$ be the greatest integer function define for $t \in \mathbf{R}$

$$k_l(t) = \left[\frac{t}{t_l}\right] \quad \text{so that} \quad \lim_{l\to\infty} t_l k_l(t) = t.$$

Since $k_l(t)$ is an integar and $\exp(t_l X_l) \in H$,

$$\exp(k_l(t) t_l X_l) = (\exp(t_l X_l))^{k_l(t)} \in H.$$

But $\lim_{l\to\infty} k_l(t) X_l = tX$ and as H is closed and exp continuous we have $\exp(tX) = \lim_{l\to\infty} \exp(k_l(t) X_l) \in H$. This completes the proof. □

LEMMA 2.3.4. *Let $\mathfrak{h}$ be the subset of the Lie algebra $\mathfrak{g}$ of G defined by $\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \text{ for all } t\}$. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.*

PROOF. We first show $\mathfrak{h}$ is closed under sums. Let $X, Y \in \mathfrak{h}$. Then $\exp(tX)\exp(tY) \in H$ for all $t \in \mathbf{R}$. But

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + tZ_t) \tag{2.2}$$

where $\lim_{t\to 0} Z_t = 0$. Taking any sequence of positive numbers $\{t_l\}$ so that $\lim_{l\to\infty} t_l = 0$ and setting $X_l := X + Y + t_l Z_{t_l}$ we can use Lemma 2.3.3 to conclude that $X + Y \in \mathfrak{h}$.

We now need to show h is closed under Lie bracket. If $X, Y \in \mathfrak{h}$ then for all $t \in \mathbf{R}$ there is a $W_t \in \mathfrak{g}$ so that

$$\exp(tX)\exp(tY)\exp(tX)^{-1}\exp(tY)^{-1} = \exp(t^2[X,Y] + t^2 W_t) \in H \tag{2.3}$$

and $\lim_{t\to 0} W_t = 0$. So that anther application of Lemma 2.3.3 implies $[X,Y] \in \mathfrak{h}$. This completes the proof. □

Lemma 2.3.5. *Using the notation of the last lemma let $\mathfrak{h}'$ be a complementary subspace to $\mathfrak{h}$ in $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$. Then there is a open neighborhood V' of 0 in $\mathfrak{h}'$ so that $0 \neq Y \in V'$ implies $\exp Y \notin H$.*

Proof. If the lemma is false there is a sequence $0 \neq Y_l \in \mathfrak{h}'$ with $\lim_{l\to\infty} Y_l = 0$ and $\exp(Y_l) \in H$. Put a Euclidean norm $\|\cdot\|$ on $\mathfrak{h}'$ and let K be the closed annulus $K := \{X \in \mathfrak{h}' : 1 \leq \|X\| \leq 2\}$. We can assume that $\|Y_l\| \leq 1$ for all l which implies there are integers n_l so that $n_l Y_l \in K$. Since K is compact by going to a subsequence we can assume that $\lim_{l\to\infty} n_l Y_l = X$ for some $X \in K$. Then if we set $X_l := n_l Y_l$ and $t_l := 1/n_l$ then Lemma 2.3.3 implies $\exp(tX) \in H$ for all t and thus $X \in \mathfrak{h}$. But then $X \in \mathfrak{h} \cap \mathfrak{h}' = \{0\}$ which contradicts that $\|X\| \geq 0$. This completes the proof. □

Lemma 2.3.6. *There is an open neighborhood U of the identity e of G so that $U \cap H$ is a smooth submanifold of U.*

Proof. Let V' be the neighborhood of 0 in $\mathfrak{h}'$ given by the last lemma and let V be a small open neighborhood of 0 in $\mathfrak{h}$. Then by making V and V' smaller we can assume the map $(X, X') \mapsto \exp(X)\exp(X')$ from the open neighborhood $V \times V'$ in $\mathfrak{g}$ is a diffeomorphism onto an open neighborhood U of e in G. Assume that $\exp(X)\exp(X') \in H$. Then as $X \in V \subset \mathfrak{h}$ we have $\exp(X) \in H$ so that also $\exp(X') \in H$. By Lemma 2.3.5 this implies that $X' = 0$. Thus $H \cap U = \{\exp(X) : X \in V\}$ which is a smooth (and in fact real analytic) submanifold of U. This completes the proof. □

Proof of Theorem 2.3.1. Let U be the neighborhood of the identity e of G given by the last lemma. Then for any point $\xi \in G$ the open neighborhood $\xi U := \{\xi g : g \in U\}$ is an open neighborhood of ξ in G so that $\xi U \cap H$ is a submanifold of ξU. Therefore H is an embedded submanifold of G. That H is a Lie subgroup is now straightforward. □

A good deal of both integral geometry and harmonic analysis on homogeneous spaces involves integration over spaces such as the space of all lines or all planes in in $\mathbf{R}^3$, the space of all circles on the sphere S^2 and the like where what these examples have in common is that they have a transitive action by a Lie group. We will show that under very minimal hypothesis this implies the object in question must be a smooth manifold and can be realized as a "homogeneous space" or coset space G/H for Lie group G with closed subgroup H. This point of view is important not only because it lets us see that most objects in mathematics that have a transitive continuous group of symmetries are "nice" in the sense they are manifolds, the realization of these spaces as homogeneous spaces G/H makes it possible to deal with the analysis and geometry of these spaces in a uniform manner.

Here is the basic outline of how this works. Let X be a set and let G be an abstract group. By an ***action of** G **on** X* we mean a map $(g, x) \mapsto gx$ from $G \times X \to X$ so that $ex = x$ for all $x \in X$ and $(g_1 g_2)x = g_1(g_2 x)$. This implies that $g^{-1}gx = x$ for all g and x and therefore the map $x \mapsto gx$ is invertible with inverse $x \mapsto gx$. The action is *transitive* iff for all $x_1, x_2 \in X$ there is a $g \in G$ with $gx_1 = x_2$. That is any element of X can be moved to any other element by a member of G. Let $x_0 \in X$ and let $H = \{a \in G : ax_0 = x_0\}$. The subgroup H is the ***one point stabilizer of*** x_0 or the ***isotropy subgroup of*** G ***at*** x_0. Let $G/H = \{\xi H : \xi \in G\}$ be the space of left cosets of H in G. Note there is also a natural action of G on the space G/H given by $g(\xi H) := (g\xi)H$.

EXERCISE 2.3.7. Assume that the action of G on X is transitive and let H be the one point stabilizer of x_0 in G. Then show:

1. The map $\varphi : G/H \to X$ given by $\varphi(\xi H) = \xi x_0$ is a bijection of G/H with X. This map commutes with the action of G, that is $\varphi(g\xi H) = g\varphi(\xi H)$.
2. If $x_1 \in X$ let $H_1 = \{a \in G : ax_1 = x_1\}$, then $H_1 = gHg^{-1}$ where g is any element of G with $gx_0 = x_1$.
3. An element ξ fixes every point of X (i.e. $\xi x = x$ all $x \in X$) if and $\xi \in \bigcap_{g \in G} gHg^{-1}$. □

We now set up some notation. Let H be a closed subgroup of the Lie group G with $\dim G = n$ and $\dim H = k$ and let G/H be the set of left cosets of H in G. Let $\pi : G \to G/H$ be the natural projection $\pi(\xi) = \xi H$. This maps commutes with the action of G, $\pi(g\xi) = g\pi(\xi)$. Give G/H the quotient topology, that is a subset $U \subseteq G/H$ is open iff the preimage $\pi^{-1}[U]$ is open in G. This maps clearly maps π continuous. It also makes π into an open map, that is if $V \subseteq G$ is open, then $\pi[V]$ is open in G/H. This is because $\pi^{-1}[\pi V] = VH := \bigcup_{a \in H} Va$ is a union of open sets and thus open.

THEOREM 2.3.8. *If G is a Lie group and H a closed subgroup of G, then, with the topology above, the space G/H has a natural structure of a smooth manifold of dimension* $\dim G - \dim H$.

PROOF. The idea of the proof is simple and natural. Choose a submanifold of G of dimension $\dim G - \dim H$ that is transverse to each of the cosets ξH it meets and so that it only meets each coset at most once. Then coordinates on the submanifold give coordinates on the space of cosets. What takes some work is showing that the transition functions between coordinates constructed in this way are smooth.

We first show that G/H is a Hausdorff space. If $\xi_0, \eta_0 \in G$ with $\xi_0 H \neq \xi_1 H$ (so $\pi(\xi_0) \neq \pi(\xi_1)$ in G/H), then there are open neighborhoods V_0 and

V_1 of ξ_0 and ξ_1 so that $V_0H \cap V_1H = \varnothing$. To see this note $\xi_1^{-1}\xi_0 \notin H$ and H is closed. Thus by the continuity of the map $(\xi, \eta) \mapsto \eta^{-1}\xi$ there are open neighborhoods V_i of ξ_i so that the set $\{\eta^{-1}\xi : \xi \in V_1, \eta \in V_2\}$ is disjoint form H. From this it is not hard to check that V_1H and V_2H are disjoint. As each V_iH is a union of the open sets V_ia with $a \in H$ it is an open set. As the map π is open the sets $\pi[V_0H]$ and $\pi[V_1H]$ are disjoint open sets in G/H and as $\pi(\xi_i) \in \pi[V_iH]$ this shows distinct points of G/H have disjoint neighborhoods so G/H is Hausdorff.

Let $n = \dim G$ and $k = \dim H$. Then call a submanifold M of G ***nicely transverse*** iff it has dimension $n - k$, at each $\xi \in M$ the submanifolds M and ξH intersect transversely at ξ (that is $T(M)_\xi \cap T(\xi H)_\xi = \{0\}$) and finally the set ξH only intersects M at the one point ξ. Let N be anther nicely transverse submanifold of G. Let U be the subset of M of points ξ so that the set ξH meets the submanifold N in some point which we denote by $\varphi(\xi)$. We now claim that U is open in M, $\varphi[U]$ is open in N and that the map $\varphi : U \to \varphi[U]$ is a smooth diffeomorphism. Define a function $f : M \times H \to G$ by $f(\xi, a) := \xi a$. Let $X_1, \dots, X_{n-k}$ be a basis for $T(M)_\xi$ and $Y_1, \dots, Y_k$ a basis of $T(H)_a$. Then using the formulas of proposition 2.1.5

$$f_{*(\xi,a)}(X_i, 0) = R_{a*}X_1, \quad f_{*(\xi,a)}(0, Y_j) = L_{\xi*}Y_j.$$

Note that $R_{*a}X_1, \dots, R_{*a}X_{n-k}$ are linearly independent modulo the subspace $T(\xi H)_{\xi a}$ (as $X_1, \dots, X_{n-k}$ are linearly independent modulo $T(\xi H)_\xi$ and R_{*a} maps $T(\xi H)_\xi$ onto $T(\xi H)_{\xi a}$). Also $L_{\xi*}$ maps $Y_1, \dots, Y_k$ onto a basis of $T(\xi H)_{\xi a}$. It follows that $f_{*(\xi,a)}$ maps a basis of $T(M \times H)_{(\xi,a)}$ onto a basis of $T(G)_{\xi a}$. Therefore by the inverse function theorem the map f is a local diffeomorphism. But the hypothesis that M is a nicely transverse implies that f is injective. (If $\xi_1 a_1 = \xi_2 a_2$, then $\xi_1 = \xi_1 a_2 a_1^{-1}$ and as the orbit $\xi_1 H$ only meets M at ξ_1 this implies $\xi_1 = \xi_2$ and $a_1 = a_2$.) Therefore is a diffeomorphism of $M \times H$ onto the open set $f[M \times H] = MH = \{\xi a : \xi \in M, a \in H\}$. As the set $\varphi[U]$ is just the intersection of N with MH it follows that $\varphi[U]$ is open in N. A similar argument replacing, but reversing the roles of M and N, shows that U is open in M.

Any point of $\varphi[U]$ can be written uniquely as $f(\xi, a) = \xi a$ for $\xi \in U$ and $a \in H$. The inverse of the map φ is then given by $f(\xi, a) \mapsto (\xi, a)$. As the level sets $\{f(\xi_0, a) : a \in H\} = \xi_0 H$ are all transverse to N the implicit function theorem implies this maps is smooth. Thus the inverse of φ is smooth. Again a similar argument reversing the roles of M and N shows that φ is smooth. Thus $\varphi : U \to \varphi[U]$ is a diffeomorphism as claimed.

We now construct coordinates on G/H. Let $x_0 \in G/H$. Choose $\xi_0 \in G$ with $\pi(\xi_0) = x_0$. Then there is a nicely transverse submanifold M with $\xi_0 \in M$. By making M a little smaller we can assume that there is a diffeomorphism $u_M : M \to V_M$ where $V_M \subseteq \mathbf{R}^{n-k}$ is an open set. As above the set MH is open in G and thus $\pi[M] = \pi[MH]$ is open in G/H and the restriction of π to M is a bijection of M with $\pi[M]$. Define $v_M : \pi[M] \to V_M$ by $v_M = \pi|_M^{-1} \circ u_M$. The function v_M thus gives local coordinates on the open set $\pi[M]$. To see this defines a smooth structure on G/H we need to check that the transition functions between coordinates are smooth. Let N be anther nicely transverse submanifold of G and let v_N be the coordinates function defined on $\pi[N]$. Let $U \subseteq M$ be as above and let $\varphi : U \to \varphi[U]$ be as above. Then the transition function $\tau_{M,N} : v_M[V_M \cap V_N] \to v_N[V_M \cap V_N]$ is given by

$$\tau_{M,N} = v_M^{-1} \circ v_N = u_M^{-1} \circ \varphi \circ u_N$$

which is clearly smooth. This completes the proof. □

2.3.2. Invariant Volume Forms. We are now interested in when the homogeneous space G/H has an invariant volume form. There is a easy necessary and sufficient condition for the existence of such a form, but first we need a little notation. Let G be Lie group of dimension n and let H be a closed subgroup of G dimension k. Then there is a linearly independent set of left invariant one forms $\omega^1, \dots, \omega^{n-k}$ so that the restriction of each ω^i to $T(H)_e$ is zero. By left invariance this implies that the restriction of each ω^i to $T(\xi H)_\xi$ is zero for each ξ. Thus for each ξ the $T(\xi H)_\xi = \{X \in T(G)_\xi : \omega^1(X) = \omega^{n-k}(X) = 0\}$. If $\sigma^1, \dots, \sigma^{n-k}$ is anther such set of left invariant one forms, then the is a nonsingular matrix c^i_j so that $\sigma^i = \sum_j c^i_j \omega^j$. This implies the $(n-k)$-form

$$\omega_{G/H} := \omega^1 \wedge \cdots \wedge \omega^{n-k}$$

is well defined up to a nonzero constant multiple.

THEOREM 2.3.9. *The homogeneous space G/H has a G invariant volume form $\Omega_{G/H}$ if an only if the form $\omega_{G/H}$ is closed (i.e. $d\omega_{G/H} = 0$). If this holds, then $\omega_{G/H} = \pi^*\Omega_{G/H}$ where $\pi : G \to G/H$ is the natural projection.*

REMARK 2.3.10. This is from the book [**23**] of Santaló page 166. For other conditions that imply the existence of an invariant measure see [**23**, p. 168, and §10.3 pp.170–173].

PROOF. If G/H has an invariant volume form $\Omega_{G/H}$, then $\pi^*\Omega_{G/H}$ is a left invariant $(n-k)$-form on G so that $\iota_X \pi^*\Omega_{G/H} = 0$ for all $X \in T(H)_e$. This, and a little linear algebra, show that $\omega_{G/H} = c\pi^*\Omega_{G/H}$ for some constant c. Thus $d\omega_{G/H} = cd\pi^*\Omega_{G/H} = c\pi^* d\Omega_{G/H} = 0$ as $d\Omega_{G/H} = 0$ for reasons of dimension.

Now assume that $d\omega_{G/H} = 0$. We first claim that $\omega_{G/H} = \pi^*\Omega$ for a unique form Ω on G/H. To see this let $x_0 \in G/H$ and choose coordinates $x^1, \dots, x^{n-k}$ centered at x_0 (where $n = \dim G$, $k = \dim H$). Let ξ_0 be a point in G with $\pi(\xi_0) = x_0$. Define functions $u^1, \dots, u^{n-k}$ near ξ_0 by $u^i = \pi^* x^i = x^i \circ \pi$. Then by the implicit function theorem there are function $y^1, \dots, y^k$ so that $y^1, \dots, y^k, u^1, \dots, u^{n-k}$ are coordinates on G centered at ξ_0. Note that the forms $du^1, \dots, du^{n-k}$ all vanish on all of the vector tangent to a fiber $\pi^{-1}[x]$ and thus locally are in the span (over the smooth functions on G) of the forms $\omega^1, \dots, \omega^{n-k}$. It follows that in the coordinates $y^1, \dots, y^k, u_1, \dots, u^{n-k}$ the form $\omega_{G/H}$ is of the form

$$\omega_{G/H} = a(y^i, u^l) du^1 \wedge \cdots \wedge du^{n-k}$$

for a unique smooth function $a(y^i, u^l)$. Then

$$0 = d\omega_{G/H} = \sum_{l=1}^{k} dy^l \frac{\partial a}{\partial y^l} dy^l \wedge du^1, \dots, \wedge du^{n-k}.$$

This implies $\partial a/\partial y^l = 0$ for all $l = 1, \dots, k$ and thus that a is independent of $y^1, \dots, y^k$, so $a = a(u^1, \dots, u^{n-k})$ and

$$\omega_{G/H} = a(u^1, \dots, u^{n-k}) du^1 \wedge \cdots \wedge du^{n-k} = \pi^* \Omega$$
$$\Omega = a(x^1, \dots, x^{n-k}) dx^1 \wedge \cdots \wedge dx^{n-k}.$$

This clearly uniquely defines Ω near x_0 and the uniqueness shows that Ω is globally defined on G/H. But then $\pi^* g^* \Omega = L_g^* \pi^* \Omega = L_g^* \Omega_{G/H} = \Omega_{G/H} = \pi^* \Omega$. As π is a submersion this yields $g^* \Omega = \Omega$ and so G/H has the invariant volume form $\Omega_{G/H} = \Omega$. □

PROPOSITION 2.3.11. *If ω is a left invariant form and $g \in G$, then the forms $d\omega$ and $R_g^* \omega$ are also left invariant.*

PROOF. If ω is left invariant, then $L_g^* d\omega = dL_g^* \omega = d\omega$ and so $d\omega$ is also left invariant. As the maps R_g and L_{g_1} commute for all $g_1, g \in G$ we have $L_{g_1}^* R_g^* \omega = R_g^* L_{g_1}^* \omega = R_g^* \omega$, which shows that $R_g^* \omega$ is left invariant. □

2.3.3. Invariant Riemannian Metrics. Let G be a Lie group and K a closed subgroup of G. The geometry of the homogeneous G/K is easier to understand if it is possible to put a Riemannian metric on G/K that is invariant under the action of G on G/K. One reason for this is that it is often useful to have a metric space structure on G/K that is invariant by the action of G and an invariant Riemannian gives such a structure. If the group K is compact then we can use a standard averaging trick to show that G/K has such a metric:

THEOREM 2.3.12. *Let G be a Lie group and K a compact subgroup of G. Then the homogeneous space G/K has an invariant Riemannian metric. Taking $K = \{e\}$ shows that the group G has a left invariant Riemannian metric.*

PROOF. Let $\pi : G \to G/K$ be the natural projection and let $\mathbf{o} = \pi(e)$ be the origin of G/K. Then the group K acts the tangent space $T(G/K)_{\mathbf{o}}$ by the action $a \cdot X = a_* X$ where a_* is the derivative of a at $\mathbf{o}$. Let $g_0(\, ,)$ be any positive definite inner product on the vector space $T(G/K)_{\mathbf{o}}$. As the group K is compact it has a bi-invariant measure da. Then define a new inner product $g(\,,\,)$ on $T(G/K)_{\mathbf{o}}$ by

$$g(X,Y)=\int_K g_0(a_*X,a_*Y)\,da.$$

This in invariant under the action of K: If $b\in K$ then (using the change of variable $a\mapsto ab^{-1}$)

$$g(b_*X,b_*Y)=\int_K g_0(a_*b_*X,a_*b_*Y)\,da=\int_K g_0(a_*X,a_*Y)\,da=g(X,Y).$$

Define a Riemannian metric $\langle\,,\rangle$ on G/K by choosing for each $x\in G/K$ an element $\xi\in G$ with $\xi\mathbf{o}=x$ and setting

$$\langle X,Y\rangle_x=g(\xi_*^{-1}X,\xi_*^{-1}Y). \tag{2.4}$$

This is independent of the choice of ξ with $\xi\mathbf{o}=x$ for if $\xi'\mathbf{o}=x$ then $\xi'=\xi a$ for some $a\in K$ and therefore

$$g((\xi')_*^{-1}X,(\xi')_*^{-1}Y)=g(a_*^{-1}\xi_*^{-1}X,a_*^{-1}\xi_*^{-1}Y)=g(\xi_*^{-1}X,\xi_*^{-1}Y)$$

by the invariance of $g(\,,\,)$ under K. Finally if $x_0\in G/K$ then as the map $\pi:G\to G/K$ is a submersion there is neighborhood U of x_0 and a smooth function $\xi:U\to G$ so that $\pi(\xi(x))=x$ for all $x\in U$, that is $\xi(x)\mathbf{o}=\pi(\xi(\mathbf{o}))=x$. This implies that near any point x_0 of G/K it is possible to choose the elements ξ in the definition (2.4) to depend smoothly on x. Thus $\langle\,,\rangle$ is a smooth Riemannian metric on G/K. We leave showing that this metric is invariant under G as an exercise. $\square$

COROLLARY 2.3.13. *If G/K is a homogeneous space with K compact then G/K has an a measure invariant under G.*

PROOF. The space G/K has an invariant Riemannian metric and thus the Riemannian volume measure is invariant under the action of G. $\square$

It will often be useful to have a left invariant Riemannian metric on G that related in a nice way to a given invariant Riemannian on G/K.

PROPOSITION 2.3.14. *Let G be a Lie group and K a closed subgroup of G. Let $g(\,,\,)$ be a Riemannian metric on G which is left invariant under elements of G and also right invariant under elements of K. Then there is a unique Riemannian metric $\langle\,,\rangle$ on G/K so that the natural map $\pi:G\to G/K$ is a Riemannian submersion. This metric is invariant under the action of G on G/K.*

*Conversely if K is compact and $\langle\,,\rangle$ is an invariant Riemannian on metric on G/K then there is a Riemannian metric g on G which is left invariant under all elements of G and right invariant under elements of K. We will say that the metric $g(\,,\,)$ is **adapted** to the metric $\langle\,,\rangle$.*

PROOF. A (tedious) exercise in chasing through definitions. (In proving the section part it is necessary to average over the subgroup K to insure that the metric $g(\, , \,)$ is right invariant under elements of K.) $\square$

PROPOSITION 2.3.15. *Let G/K be a homogeneous space with K compact. Let $\langle \, , \rangle$ be an invariant Riemannian metric on G/K and assume G has a Riemannian metric that is adapted to $\langle \, , \rangle$ in the sense of the last proposition. Then for any integrable function on f on G/K*

$$\operatorname{Vol}(K) \int_{G/K} f(x)\, dx = \int_G f(\pi\xi)\, d\xi \tag{2.5}$$

where $d\xi$ is the Riemannian measure on G and $\operatorname{Vol}(K)$ is the volume of K as a Riemannian submanifold of G.

Likewise if h is an integrable function on G then

$$\int_G h(\eta)\, d\eta = \int_{G/K} \int_{\pi^{-1}[x]} h(\xi)\, d\xi\, dx \tag{2.6}$$

where $d\xi$ is the volume measure of $\pi^{-1}[x]$ considered as a Riemannian submanifold of G.

PROOF. A straight forward exercise in the use of the coarea formula. $\square$

2.3.4. Invariant Forms on Matrix Groups. Many if not most of the Lie groups encountered in geometry are matrix groups. Fortunately they are in several ways easier to deal with than general Lie groups. In particular there are several methods for finding the left and right invariant on matrix groups and their homogeneous spaces. As a first example of this we let $GL(n, \mathbf{R})$ be the general linear group over the reals. That is $GL(n, \mathbf{R})$ is the group of $n \times n$-matrices with non-zero determinant. We use the natural coordinates $X = [x_i^j]$ on $GL(n, \mathbf{R})$. It terms of these coordinates the following lets us find the left and right invariant forms.

PROPOSITION 2.3.16. *If $X = [x_i^j]$ is matrix of coordinate functions on $GL(n, \mathbf{R})$, then the elements of the matrix $X^{-1}dX$ are a basis of the left invariant one forms of $GL(n, \mathbf{R})$. If G is a Lie subgroup of $GL(n, \mathbf{R})$ of dimension m, then the basis of the left invariant one forms on G can be found by restricting the some collection of invariant one forms of $GL(m, \mathbf{R})$ down to G. Likewise the elements of the matrix $(dX)X^{-1}$ gives a basis of the right invariant one forms on $GL(n, \mathbf{R})$ and by restriction these can be used to find the right invariant one forms on any Lie subgroup of $GL(n, \mathbf{R})$.*

PROOF. Let $A \in GL(n, \mathbf{R})$ be a constant matrix. Then left translation by A is matrix multiplication on the left by A: $L_A(X) = AX$. As A is constant $d(AX) = AdX$. Thus

$$L_A^*(X^{-1}dX) = (AX)^{-1}d(AX) = X^{-1}A^{-1}AdX = X^{-1}dX.$$

Thus the elements of $X^{-1}dX$ are left invariant as claimed. At the identity matrix I we have $(X^{-1}dX)_I = [dx_i^j]_I$ and these are linearly independent. Thus the elements of $X^{-1}dX$ form a basis of the left invariant one forms as claimed. That the left invariant one forms on a Lie subgroups can be found by restriction is straight forward linear algebra and left to the reader. The proof in the case or right invariant forms is identical. $\square$

We now give several examples of this. As a first example let

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x, y \in \mathbf{R}, x \neq 0 \right\}.$$

This is the group of all affine mappings of the line $\mathbf{R}$. Letting $g = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$ we have

$$g^{-1} = \begin{bmatrix} \frac{1}{x} & \frac{-y}{x} \\ 0 & 1 \end{bmatrix}, \quad dg = \begin{bmatrix} dx & dy \\ 0 & 0 \end{bmatrix}$$

$$g^{-1}dg = \begin{bmatrix} \frac{dx}{x} & \frac{dy}{x} \\ 0 & 0 \end{bmatrix}, \quad dgg^{-1} = \begin{bmatrix} \frac{dx}{x} & \frac{-y\,dx}{x} + dy \\ 0 & 0 \end{bmatrix}$$

Thus the elements dx/x and dy/x of $g^{-1}dg$ give a basis for the left invariant one forms on G and the elements dx/x and $-(y\,dx)/x + dy$ of dgg^{-1} are a basis of the right invariant one forms on G. This implies

$$\Omega_G = \frac{dx \wedge dy}{x^2}$$

is a left in variant volume form on G and

$$\Theta = \frac{dx \wedge dy}{x}$$

is a right invariant volume form. The relation $\Theta = \Delta_G^+ \Omega_G$ of proposition 2.2.2 then implies

$$\Delta_G^+(x) = \frac{1}{x}.$$

Thus G is not unimodular and this shows the function Δ_G^+ can change sign as claimed above.

For this group we now give some homogeneous spaces and use the theory above to investigate if they have an invariant volume form. First identify the real numbers $\mathbf{R}$ with the set of column vectors of the form $\begin{bmatrix} a \\ 1 \end{bmatrix}$. Then G acts on $\mathbf{R}$ by left multiplication $\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} = \begin{bmatrix} xa + y \\ 1 \end{bmatrix}$. The subgroup H fixing the element $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

$$H = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} : x \in \mathbf{R}, x \neq 0 \right\}.$$

The form $\omega_{G/H}$ is given by

$$\omega_{G/H} = \frac{dy}{x}.$$

As $d\omega_{G/H} = -(dx \wedge dy)/x^2 \neq 0$ this implies the homogeneous space $G/H = \mathbf{R}$ has no invariant volume form invariant under G. (While it is clear that $\mathbf{R}$ has no measure invariant under the group of affine maps and the above may seem like over kill it is useful to see how the theory works in easy to understand cases before applying it it cases where the results are not obvious.)

The group also acts on $\mathbf{R}^{\#}$, the space of nonzero real numbers, by $ga = xa$. In this case the subgroup fixing the point 1 is

$$H = \left\{ \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} : y \in \mathbf{R} \right\}.$$

So in this case $\omega_{G/H} = dx/x$ and this a closed. Thus $\mathbf{R}^{\#}$ has a G invariant volume form, and if x is the natural coordinate on $\mathbf{R}$, then dx/x is the invariant volume form on $\mathbf{R}^{\#}$. Note that dx/x is also the invariant volume form on $\mathbf{R}^{\#}$ considered as a multiplicative group.

We now look at a more interesting example. Let $\mathbf{E}(2)$ be the group of rigid orientation preserving motions of the plane $\mathbf{R}^2$. If we identify $\mathbf{R}^2$ with the space of column vectors $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$ then the group $\mathbf{E}(2)$ can be realized as a the matrix group

$$\mathbf{E}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} : 0 \leq \theta < 2\pi, x, y \in \mathbf{R} \right\}$$

Letting $g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$ we have

$$g^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & -x\cos\theta - y\sin\theta \\ -\sin\theta & \cos\theta & x\sin\theta - y\cos\theta \\ 0 & 0 & 1 \end{bmatrix},$$

$$dg = \begin{bmatrix} -\sin\theta\, d\theta & -\cos\theta\, d\theta & dx \\ \cos\theta\, d\theta & -\sin\theta\, d\theta & dy \\ 0 & 0 & 0 \end{bmatrix},$$

$$g^{-1}dg = \begin{bmatrix} 0 & -d\theta & \cos\theta\, dx + \sin\theta\, dy \\ d\theta & 0 & -\sin\theta\, dx + \cos\theta\, dy \\ 0 & 0 & 0 \end{bmatrix},$$

$$dgg^{-1} = \begin{bmatrix} 0 & -d\theta & y\, d\theta + dx \\ d\theta & 0 & -x\, d\theta + dy \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus a basis for the left invariant one forms is $d\theta$, $\cos\theta\,dx + \sin\theta\,dy$ and $-\sin\theta\,dx + \cos\theta\,dy$. A basis for the right invariant one forms is $d\theta$, $y\,d\theta + dx$ and $-x\,d\theta + dy$. Taking the exterior product of these elements yields that

$$\Omega_{\mathbf{E}(2)} = d\theta \wedge dx \wedge dy = dx \wedge dy \wedge d\theta$$

is a bi-invariant volume form on $\mathbf{E}(2)$.

The group $\mathbf{E}(2)$ has a transitive action $\mathbf{R}^2$. The subgroup fixing the origin is

$$SO(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : 0 \le \theta < 2\pi \right\}$$

The foliation of $\mathbf{E}(2)$ by the left cosets of $SO(2)$ is defined by $\cos\theta\,dx + \sin\theta\,dy = -\sin\theta\,dx + \cos\theta\,dy = 0$. Therefore

$$\omega_{\mathbf{R}^2} = (\cos\theta\,dx + \sin\theta\,dy) \wedge (-\sin\theta\,dx + \cos\theta\,dy) = dx \wedge dy.$$

This is closed, so the space $\mathbf{R}^2$ has the invariant area form $dx \wedge dy$. Of course we knew this was an invariant volume before starting the calculation. The next example is less obvious.

For a much more interesting example let $AG(1,2)$ be the space of all oriented lines in $\mathbf{R}^2$. That is a straight line together with a choice of one of the two directions along the line. The group $\mathbf{E}(2)$ is transitive on the set $AG(1,2)$ and thus $AG(1,2)$ is a homogeneous space for $\mathbf{E}(2)$. Let L_0 the x-axis with its usual orientation. Then the subgroup of elements of $\mathbf{E}(2)$ fixing L_0 is

$$H = \left\{ \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \mathbf{R} \right\}.$$

The foliation of $\mathbf{E}(2)$ by left cosets of H is defined by $d\theta = -\sin\theta\,dx + \cos\theta\,dy = 0$. Thus

$$\omega_{AG(1,2)} = (-\sin\theta\,dx + \cos\theta\,dy) \wedge d\theta.$$

This form is closed so $AG(1,2)$ has an invariant area form. To get a more usable form of it. If $g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$ and L_0 is the x-axis, then the direction of the line gL_0 is the vector $(\cos\theta, \sin\theta)$ and thus the normal vector to gL_0 is $(-\cos\theta, \sin\theta)$. Also the point (x, y) is on the line gL_0 thus the distance of the line gL_0 to the origin is

$$p = (x, y) \cdot (-\sin\theta, \cos\theta) = -x\sin\theta + y\cos\theta$$

A calculation shows

$$dp \wedge d\theta = (-\sin\theta\,dx + \cos\theta\,dy) \wedge d\theta = \omega_{AG(1,2)}.$$

Thus $\Omega_{AG(1,2)} = dp \wedge d\theta$ is an invariant area form on $AG(1,2)$. Note this in this example the isotropy subgroup H is not compact and the space $AG(1,2)$ does not have an invariant Riemannian metric.

CHAPTER 3

Representations, Submodules, Characters and the Convolution Algebra of a Homogeneous Space

3.1. Representations and Characters

Let G be any group and V a vector space. Then a ***representation of G on V*** is a group homomorphism $\rho : G \to GL(V)$ where $GL(V)$ is the ***general linear group*** of V. (That is $GL(V)$ is the group of all invertible linear maps $A : V \to V$. When it is clear from context what that homomorphism ρ is, then we sometimes write $gv := \rho(g)v$. In other terminology if $\rho : G \to GL(V)$ is a representation, then V is a ***G-module*** and G is said to have an ***action*** on V. A subspace $W \subseteq V$ of the G-module V is a ***submodule*** iff $gW := \{gv : v \in W\} \subseteq W$ for all $g \in G$. (If W is a submodule then it is not hard to show that $gW = W$ for all $g \in G$.) A G-module is ***irreducible*** iff the only the only submodules of V are the trivial submodules $\{0\}$ and V. An ***irreducible representation*** is a representation $\rho : G \to GL(V)$ so that V is an irreducible G-module.

Two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are ***isomorphic*** or ***equivalent*** iff there is an invertible linear map $S : V_1 \to V_2$ so that

$$S\rho_1(g)v = \rho_2(g)Sv \quad \text{for all } g \in G \text{ and } v \in V. \tag{3.1}$$

More generally given two two representations of $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ any linear map (but not necessarily invertible) linear S that satisfies (3.1) is called an ***intertwining map***, a ***G-module homomorphism*** or often just a ***G-map***. The following result is elementary but basic to the theory.

Proposition 3.1.1 (Schur's Lemma). *Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be two representations of G and $S : V_1 \to V_2$ an intertwining map.*

1. *If V_1 is irreducible then S is either injective or the zero map.*
2. *If V_2 is irreducible the S is either surjective or the zero map.*
3. *If V_1 and V_2 are both irreducible then S is either an isomorphism or the zero map.*

Proof. As S is an intertwining map $\ker S$ is a submodule of V_1 and the image $S[V_1]$ is a submodule of V_2. If V_1 is irreducible then $\ker S = \{0\}$, in which case S is injective, or $\ker S = V_1$ in which case $S = 0$. Likewise if

V_2 is irreducible then $S[V_1] = V_2$ or $S[V_1] = \{0\}$ which proves Part 2. The third part follows from the first two. $\square$

EXERCISE 3.1.2. Let V be an irreducible G module and let $\mathbf{D}$ be the set of all linear maps $S : V \to V$ that intertwine the G-action. That is $Sgv = gSv$ for all $g \in G$ and $v \in V$. Then show that $\mathbf{D}$ is a division algebra. $\square$

REMARK 3.1.3. With the notation the last exercise, let $\mathbf{F}$ be the base field of the vector space V. (In our considerations $\mathbf{F} = \mathbf{R}$, or $\mathbf{F} = \mathbf{C}$.) Then $\mathbf{F} \subseteq \mathbf{D}$ by identifying $c \in \mathbf{F}$ with $c\,\mathrm{Id} \in \mathbf{D}$. It is known that when $\mathbf{D}$ is finite dimensional and $\mathbf{F} = \mathbf{R}$ that the only possibilities for $\mathbf{D}$ are $\mathbf{D} = \mathbf{R}$, $\mathbf{D} = \mathbf{C}$, or $\mathbf{D} = \mathbf{H}$ (the four dimensional division algebra of quaternions). Thus the real finite dimensional irreducible representations of a group G split into three types, the real representations, the complex representations, and the quaternionic representations, depending on the algebra $\mathbf{D}$. In parts of the algebraic theory this distinction is important. We will be able to ignore it. In the complex case things are simpler. In this if $\mathbf{D}$ is finite dimensional then it follows form the fundamental theory of algebra that $\mathbf{D} = \mathbf{C}$. (To see this note that if $a \in \mathbf{D}$ then $1, a, a^2, a^3, \ldots$ are linearly dependent as $\mathbf{D}$ is finite dimensional. Thus a satisfies a polynomial equation and so $a \in \mathbf{C}$).

If $\rho : G \to GL(V)$ is a finite dimensional representation of the group G the ***character*** of the representation is

$$\chi_\rho(g) = \operatorname{trace}(\rho(g)). \tag{3.2}$$

This is a function on G with values in the base field of the vector space V.

PROPOSITION 3.1.4. *If $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are two equivalent finite dimensional representations of G then $\chi_{\rho_1} = \chi_{\rho_2}$.*

PROOF. As the representations are equivalent there is a linear isomorphism $S : V_1 \to V_2$ so that $S\rho_1(g) = \rho_2(g)S$ for all G. That is $\rho_2(g) = S\rho_1(g)S^{-1}$. That is for all g the linear maps $\rho_1(g)$ and $\rho_2(g)$ are similar. But then it is a standard result form linear algebra that $\operatorname{trace}(\rho_1(g)) = \operatorname{trace}(\rho_2(g))$. $\square$

If V is a G-module and $K \subseteq G$ is a subgroup then

$$V^K := \{v \in V : av = v \text{ for all } a \in K\}.$$

is the set of elements of V invariant under the action of K.

As an example where G modules arise naturally assume that K is a closed subgroup of G and that the homogeneous space G/K has an invariant measure. For example this will always be the case if K is compact. An important special case is $K = \{e\}$ in which case $G/K = G$ and the left invariant measure on G is an invariant measure on G/K. In this case if $\mathcal{F}(G/K)$ is one of the following function spaces $C(G/K)$ (the continuous functions on G/K), $C^k(G/K)$ (the C^k functions on G/K where $0 \le k \le \infty$), or $L^p(G/K)$ (the measurable functions f on G/K so that $\int_{G/K} |f(x)|^p\, dx < \infty$ where $1 \le p < \infty$ or $p = \infty$ and $\|f\|_{L^\infty} := \operatorname{ess\,sup} |f| < \infty$). Then there is a natural action $\tau : G \to GL(\mathcal{F}(G/K)$ of G on any of these spaces given by given by

$$\tau_g f(x) := f(g^{-1}x). \tag{3.3}$$

It is easily checked that $\tau_{g_1 g_2} = \tau_{g_1}\tau_{g_2}$ and $\tau_e = \mathrm{Id}$ so this is a representation. As the measure dx is invariant under left translation by elements of G it follows that G acts by isometries of $L^p(G/K)$:

$$\|\tau_g f\|_{L^p} = \|f\|_{L^p}.$$

In the case G/K is compact one of our main goals is to show there is an orthogonal direct sum decomposition $L^2(G/K) = \bigoplus_\alpha E_\alpha$ into finite dimensional irreducible submodules E_α (with similar decompositions for the other spaces $L^p(G/K)$, $C^k(G/K)$). Thus in the compact case there are lots of finite dimensional submodules of the spaces $L^p(G/K)$. In the noncompact case finite dimensional submodules of the $L^p(G/K)$ spaces are harder to come by:

THEOREM 3.1.5. *Let G be a noncompact Lie group and let K be a compact subgroup of G. Then for $1 \le p < \infty$ and for any nonzero $f \in L^p(G/K)$ the set of translates $\{\tau_g f : g \in G\}$ is infinite dimensional. In particular $L^p(G/K)$ has no nonzero finite dimensional submodules.*

PROOF. As K is compact we can assume G/K has a G-invariant Riemannian metric $\langle\, ,\rangle$. Let $d : G/K \times G/K \to [0,\infty)$ be the distance function defined by $\langle\, ,\rangle$. If $0 \ne f \in L^p(G/K)$ then we can normalize so that $\|f\|_{L^p} = 1$. For $x \in G/K$ and $r > 0$ let $B(x,r) = \{y : d(x,y) < r\}$ be the ball of radius r about x. Denote by $\mathbf{o}$ the origin of G/K, that is the coset of the identity. As G/K is not compact it is unbounded in the d metric. Thus it is possible to choose a sequence of numbers $r_1 < r_2 < r_3 \nearrow \infty$ so that

$$\|f\|_{L^p(B(\mathbf{o},r_k))} := \left(\int_{B(\mathbf{o},r_k)} |f(x)|^p\, dx\right)^{\frac{1}{p}} \ge 1 - \frac{1}{9^k},$$

i.e.

$$\|f\|_{L^p((G/K)\setminus B(\mathbf{o},r_k))} \le \frac{1}{9^k}$$

Let $x_1 = \mathbf{o}$ and by recursion choose x_{k+1} so that $\operatorname{dist}(x_{k+1}, \{x_1, \cdots, x_k\}) > r_{k+1}$. Choose $g_k \in G$ with $g_k^{-1}\mathbf{o} = x_k$. Set $f_k(x) := \tau_{g_k} f(x) = f(g_k^{-1}x)$. Then $f_k(\mathbf{o}) = f(x_k)$. It follows from then invariance of the measure and the estimates above that for $i \ne j$

$$\|f_i\|_{L^p(B(x_j,r_j))} \le \frac{1}{9^{\min\{i,j\}}}$$

which in turn implies

$$\sum_{i\neq k}^{\infty} \|f_i\|_{L^p(B(x_k,r_k))} \leq \sum_{i=1}^{\infty} \frac{1}{9^i} = \frac{1}{10}.$$

Now assume for some set $\{k_1, \dots, k_l\}$ that $f_{k_1}, \dots, f_{k_l}$ are linear dependent. Let $c_1 f_{k_1} + \cdots + c_l f_{k_l} = 0$ be a non-trivial linear relation between the $f_{k_1}, \dots, f_{k_l}$. By reordering we can assume that $|c_1| \geq |c_i|$ for $1 \leq i \leq l$. By dividing by c_1

$$f_{k_1} = \sum_{i=2}^{l} b_i f_{k_i} \quad \text{where } |b_i| \leq 1.$$

But then

$$\|f_{k_1}\|_{L^p(B(x_{k_1},r_{k_1}))} = \|f\|_{L^p(B(\mathbf{o},r_{k_1}))} \geq 1 - \frac{1}{9^{k_1}} \geq \frac{8}{9}$$

and using $|b_i| \leq 1$ and the inequalities above

$$\begin{aligned}\|f_{k_1}\|_{L^p(B(x_{k_1},r_{k_1}))} &\leq \sum_{i=2}^{l} |b_i| \|f_{k_i}\|_{L^p(B(x_{k_1},r_{k_1}))} \\ &\leq \sum_{i\neq k_1} \|f_i\|_{L^p(B(x_{k_1},r_{k_1}))} \leq \frac{1}{10}.\end{aligned}$$

These lead to the contradiction $1/10 \geq 8/9$ which completes the proof. □

REMARK 3.1.6. The last theorem is false for $p = \infty$. For example let $G = \mathbf{R}^n$ and $K = \{0\}$ so that $G/K = \mathbf{R}^n$. Let $0 \neq a \in \mathbf{R}^n$ and set $f_a(x) = e^{\sqrt{-1}\langle x,a\rangle}$. Then $f_a \in L^\infty(\mathbf{R}^n)$ and $f_a(x+h) = e^{\sqrt{-1}\langle h,a\rangle} f_a(x)$ and thus the one dimensional space spanned by f_a is invariant under the action of $G = \mathbf{R}^n$ by translation.

3.1.1. The Regular Representation on $L^p(G/K)$. In this section G/K will be a homogeneous space with K compact so that G/K has an invariant Riemannian metric (cf. Theorem 2.3.12). This implies that G/K has an invariant volume measure, (the Riemannian volume measure). It also implies that G/K has an invariant metric space structure. That is let $d(x,y)$ be the Riemannian distance between $x, y \in G/K$, then $d(gx, gy) = d(x,y)$. For this section the existence of the invariant measure and the invariant are the important points and the results generalize to the setting of homogeneous spaces that satisfy this conditions.

Let $1 \leq p \leq \infty$ and let $L^p(G/K)$ be the usual Banach space of measurable functions on G/K so that the norms $\|f\|_{L^p} = (\int_{G/K} |f(x)|^p \, dx)^{1/p} < \infty$ for $p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup} |f|$. The ***left regular representation*** (or just the ***regular representation***) τ of G on $L^p(G/K)$ is

$$\tau_g f(x) := f(g^{-1}x).$$

PROPOSITION 3.1.7. *If the homogeneous space $L^p(G/K)$ has an invariant measure, then the regular representation of G on $L^p(G/K)$ acts by isometries for all $1 \leq p \leq \infty$. That is $\|\tau_g f\|_{L^p} = \|f\|_{L^p}$.*

PROOF. This follows form the invariance of the measure:

$$\begin{aligned}\|\tau_g f\|_{L^p}^p &= \int_{G/K} |\tau_g f(x)|^p\,dx = \int_{G/K} |f(g^{-1}x)|^p\,dx\\ &= \int_{G/K} |f(x)|^p\,dx = \|f\|_{L^p}^p\end{aligned}$$

when $1 \leq p < \infty$ with an equally straightforward proof in the case $p = \infty$. □

EXERCISE 3.1.8. Consider X to be one of the following function spaces on G/K. The bounded continuous functions with the L^∞ norm, the continuous functions that vanish at infinity (that is for each $\varepsilon > 0$ there is a compact set $C \subseteq G/K$ so that $\sup_{x \notin C} |f(x)| < \varepsilon$) with the L^∞ norm, and the space of uniformly continuous functions again with the L^∞ norm. Show that the regular representation is acts by isometries on all of these spaces. □

Let X be a Banach space with norm $\|\cdot\|_X$ and let $\rho \to GL(X)$ be a representation of G on X. This representation is ***strongly continuous*** iff for each $x \in X$ $\rho(\xi)$ is a bounded linear map the function $\xi \mapsto \rho(\xi)x$ is continuous in the norm topology.

There is anther notation of continuity of representations that at first looks more natural than strong continuity. Let X and Y be a Banach space then the ***operator norm*** of a linear operator $A : X \to Y$ is

$$\|A\|_{\mathrm{Op}} := \sup_{0 \neq x \in X} \frac{\|Ax\|_Y}{\|x\|_X}.$$

The operator norm defines a norm on the vector space of bounded linear maps form X to Y. In this section it will usually be the case that $X = Y$. If G is a Lie group and X a Banach space then $\rho : G \to GL(X)$ is a ***norm continuous representation*** iff each $\rho(\xi)$ is a bounded linear map and the map $\xi \mapsto \rho(\xi)$ is continuous in the norm topology. The following gives the correct insight as to which is the more useful notion in our setting:

EXERCISE 3.1.9. Let $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ be the unit circle realized as the real numbers modulo 2π times the integers. Define the representation $\tau_s f(x) = f(x+s)$ on $L^p(S^1)$.
(a) Show that if $1 \leq p \leq \infty$ that τ is not norm continuous.
(b) However if $1 \leq p < \infty$ then τ is strongly continuous.
(c) But τ is not even strongly continuous on $L^\infty(S^1)$. □

THEOREM 3.1.10. *If G/K is a homogeneous space with K compact and $1 \leq p < \infty$ then the regular representation τ of G on $L^p(G/K)$ is strongly continuous.*

PROOF. Let $C_0(G)$ be the set of continuous functions on G/K with compact support. The basic fact we use is that $C_0(G/K)$ is dense in $L^p(G/K)$ for $1 \leq p < \infty$ (but not dense if $p = \infty$.) If $\varphi \in C_0(G/K)$ then φ is uniformly continuous and thus $\lim_{g\to e} \tau_g\varphi \to \varphi$ uniformly so

$$\lim_{g\to e} \|\tau_g\varphi - \varphi\|_{L^p} = 0.$$

Choose a left invariant metric on G and let $d(\xi, \eta)$ be Riemannian distance with respect to this metric, so that $d(g\xi, g\eta) = d(\xi, \eta)$. Let $f \in L^p(G/K)$ and $\varepsilon > 0$. Then there is a $\varphi \in C_0(G/K)$ with

$$\|f - \varphi\|_{L^p} < \frac{\varepsilon}{3}.$$

Choose $\delta > 0$ so that if $d(e, g) < \delta$ then

$$\|\tau_g\varphi - \varphi\|_{L^p} < \frac{\varepsilon}{3}.$$

Then for g with $d(e, g) < \delta$ and using that τ_g is an isometry

$$\begin{aligned}\|f - \tau_g f\|_{L^p} &\leq \|f - \varphi\|_{L^p} + \|\varphi - \tau_g\varphi\|_{L^p} + \|\tau_g\varphi - \tau_g f\|_{L^p}\\ &= 2\|f - \varphi\|_{L^p} + \|\varphi - \tau_g\|_{L^p} < \varepsilon.\end{aligned}$$

If $d(g_1, g_2) < \delta$ then $d(e, g_1^{-1}g_2) < \delta$ so using the last inequality and that τ acts by isometries

$$\|\tau_{g_1} f - \tau_{g_2} f\|_{L^p} = \|f - \tau_{g_1^{-1}g_2} f\|_{L^p} < \varepsilon.$$

As ε was arbitrary this completes the proof. $\square$

3.2. Definitions and Basic Properties of the Convolution Algebra

Let $\mathcal{M}(G; K)$ be the set of all measurable functions $h : G/K \times G/K \to \mathbf{R}$ so that for all $x, y \in G/K$ and $g \in G$

$$h(gx, gy) = h(x, y). \tag{3.4}$$

REMARK 3.2.1. While the definition is in terms of real valued functions latter it will also be useful to deal with complex valued functions $h : G/K \times G/K \to \mathbf{C}$. All of the basic properties given here work in the complex case also.

Now define

$$C^\infty(G; K) := \{h \in \mathcal{M}(G; K) : h \in C^\infty(G/K \times G/K)\} \tag{3.5}$$

and for $1 \leq p \leq \infty$

(3.6)

$$L^p(G; K) := \left\{\mathcal{M}(G; K) : \int_{G/K} |h(x, \mathbf{o})|^p\, dx, \int_{G/K} |h(\mathbf{o}, y)|^p\, dy < \infty\right\}$$

and define a norm on $L^p(G;K)$ by

$$\|h\|_p = \max\left\{ \left(\int_{G/K} |h(x,\mathbf{o})|^p \, dx \right)^{\frac{1}{p}}, \left(\int_{G/K} |h(\mathbf{o},y)|^p \, dy \right)^{\frac{1}{p}} \right\} \tag{3.7}$$

If $x \in G/K$ then there is a $g \in G$ so that $gx = \mathbf{o}$ thus using the invariance of the measure dy under the action of G

$$\begin{aligned} \int_{G/K} |h(x,y)|^p \, dy &= \int_{G/K} |h(gx,gy)|^p \, dy \\ &= \int_{G/K} |h(\mathbf{o},gy)|^p \, dy = \int_{G/K} |h(\mathbf{o},y)|^p \, dy \end{aligned}$$

Likewise $\int_{G/K} |h(x,y)|^p \, dx = \int_{G/K} |h(x,\mathbf{o})|^p \, dx$. So the definition of $L^p(G;K)$ and the norm $\|h\|_p$ is independent of the choice of the origin $\mathbf{o}$.

EXAMPLE 3.2.2. We now give examples to show that $\int_{G/K} |h(\mathbf{o},y)|^p \, dy < \infty$ does now imply $\int_{G/K} |h(x,\mathbf{o})|^p \, dx < \infty$ for $h \in \mathcal{M}(G;K)$. Let G be any connected Lie group and let Δ_G be the modular function of G. Let $K = \{e\}$ be the trivial subgroup of G. Then $G/K = G$ in a natural way. Let $f : G \to \mathbf{R}$ be continuous. Then $h(x,y) := f(x^{-1}y)$ satisfies $h(gx,gy) = h(x,y)$ for $g \in G$. For this choice of h (using that under the change of variable $x \mapsto x^{-1}$ the left invariant measure maps by $dx \mapsto \Delta_G(x)dx$).

$$\begin{aligned} \int_G |h(e,y)|^p \, dy &= \int_G |f(y)|^p \, dy \\ \int_G |h(x,e)|^p \, dx &= \int_G |f(x^{-1})|^p \, dx \\ &= \int_G |f(x)|^p \Delta_G(x) \, dx \end{aligned}$$

If the group is not unimodular then $\Delta_G[G] \neq \{1\}$ is a multiplicative subgroup of $(0,\infty)$ and so Δ_G is unbounded on G. Form this it is not hard to show that there is a continuous function so that $\int_G |f(y)|^p \, dy < \infty$ but $\int_G |f(x)|^p \Delta_G(x) \, dx = \infty$. For this f the function $h(x,y) = f(x^{-1}y)$ gives the desired example. □

For any $h \in L^1(G;H)$ define an integral operator $T_h : L^p(G/K) \to L^p(G/K)$ by

$$T_h f(x) := \int_{G/K} h(x,y) f(y) \, dy \tag{3.8}$$

THEOREM 3.2.3 (Generalized Young's Inequality). *Let $h \in L^1(G;K)$. Then $T_h : L^p(G/K) \to L^p(G/K)$ is a bounded linear operator that satisfies*

$$\|T_h f\|_{L^p} \leq \|h\|_1 \|f\|_{L^p}. \tag{3.9}$$

Moreover this linear operator commutes with the action of G in the sense that

$$T_h \circ \tau_g = \tau_g \circ T_h \quad \textit{for all} \quad g \in G. \tag{3.10}$$

PROOF. That T_h is bounded as a linear map $L^p \to L^p$ and the bound (3.9) holds follow from Corollary A.1.3. To prove (3.10) let $f \in L^p(G/K)$ then

$$\begin{aligned}
(T_h \circ \tau_g) f(x) &= \int_{G/K} h(x,y) f(g^{-1}y)\, dy \\
&= \int_{G/K} h(x,gy) f(y)\, dy \qquad \text{(Change of variable } y \mapsto gy) \\
&= \int_{G/K} h(g^{-1}x,y) f(y)\, dy \quad (h(x,gy) = h(g^{-1}x, g^{-1}gy)) \\
&= (\tau_g \circ T_h) f(x)
\end{aligned}$$

□

REMARK 3.2.4. Let $\varphi \in L^1(\mathbf{R}^n)$ and let $h(x,y) := \varphi(x-y)$. Then $h \in L^1(\mathbf{R}^n; \{0\})$ and $\|h\|_1 = \|\varphi\|_{L^1}$. If $f \in L^p(\mathbf{R}^n)$ $(1 \le p \le \infty)$ then

$$T_h f(x) = \int_{\mathbf{R}^n} \varphi(x-y) f(y)\, dy = \varphi \star f(x)$$

where the convolution $\varphi \star f$ is defined by the integral. The last theorem then implies $\|\varphi \star f\|_{L^p} \le \|\varphi\|_{L^1} \|f\|_{L^p}$. This is the classical form of Young's inequality. See Exercise 3.2.8 for the extension to other groups.

For $h, k \in L^1(G;K)$ define a product $h * k$ by

$$h * k(x,y) := \int_{G/K} h(x,z) k(z,y)\, dz. \tag{3.11}$$

THEOREM 3.2.5. *The space $L^1(G;K)$ is closed under the product $(h,k) \mapsto h * k$ and*

$$\|h * k\|_1 \le \|h\|_1 \|k\|_1. \tag{3.12}$$

If $h, k \in L^1(G;K)$ then

$$T_{h*k} = T_h \circ T_k \tag{3.13}$$

where T_h is defined by (3.8) above. Thus the product $*$ is associative. Therefore $(L^1(G;K), *)$ is a Banach algebra, the ***convolution algebra*** *of G/K.*

REMARK 3.2.6. A very short history of convolutions in analysis can be found in the Hewitt and Ross [**18**, pp. 281–283]. In the setting of analysis on locally compact groups the basic papers seem[1] to be those of Weyl and Peter [**29**], Weil [**28**], and Gel'fand [**14**].

[1]I have only looked at secondary sources so these opinions should not be taken too seriously

PROOF. First note

$$\begin{aligned}
\int_{G/K} |h * k(x,\mathbf{o})|\,dx &\leq \int_{G/K}\int_{G/K} |h(x,z)||k(z,\mathbf{o})|\,dz\,dx \\
&= \int_{G/K}\int_{G/K} |h(x,z)|\,dx \int_{G/K} |k(z,y)|\,dz \\
&\leq \|h\|_1 \int_{G/K} |k(z,\mathbf{o})|\,dz \\
&\leq \|h\|_1\|k\|_1
\end{aligned}$$

and a similar calculation shows $\int_{G/K} |h * k(\mathbf{o},y)|\,dy \leq \|h\|_1\|k\|_1$. For any $g \in G$

$$\begin{aligned}
h * k(gx,gy) &= \int_{G/K} h(gx,z)k(z,gy)\,dz \\
&= \int_{G/K} h(gx,gz)h(gz,gy)\,dz \quad \text{(change of variable } z \mapsto gz\text{)} \\
&= \int_{G/K} h(x,z)k(z,y)\,dz \\
&= h * k(x,y)
\end{aligned}$$

Therefore $h * k \in L^1(G;K)$ as claimed. To get the formula for $T_h \circ T_k$ compute:

$$\begin{aligned}
(T_h \circ T_k)f(x) &= \int_{G/K} h(x,z)T_k f(z)\,dz \\
&= \int_{G/K} h(x,z)\int_{G/K} k(z,y)\,dy\,dz \\
&= \int_{G/K}\int_{G/K} h(x,z)k(z,y)\,dz f(y)\,dy \\
&= \int_{G/K} h * k(x,y)f(y)\,dy \\
&= T_{h*k}f(x).
\end{aligned}$$

As composition of maps is associative $T_{(h*k)*p} = (T_h \circ T_k)\circ T_p = T_h \circ (T_k \circ T_p) = T_{h*(k*p)}$. So associativity of $*$ follows form:

LEMMA 3.2.7. *Let $h \in L^1(G;K)$ be so that $T_h f = 0$ for all smooth f on G/K with compact support. Then $h = 0$ almost everywhere as a function on $G/K \times G/K$.*

PROOF. Let $\varphi(x,y) := \sum_{i=1}^{l} f_i(x)p_i(y)$ where f_i and p_i are smooth compactly supported functions on G/K. Then

$$\begin{aligned}
\int\int_{G/K\times G/K} h(x,y)\varphi(x,y)\,dx\,dy &= \sum_{i=1}^{l}\int_{G/K}\int_{G/K} h(x,y)f_i(x)\,dx p_i(y)\,dy \\
&= \sum_{i=1}^{l}\int_{G/K} T_h f_i(y)p_i(y)\,dy \\
&= 0
\end{aligned}$$

as $T_h f_i = 0$ for each i. But the set of functions $\sum_{l=1}^{l} f_i(x)p_i(y)$ is dense in the uniform norm in the set of all continuous functions with compact support. Thus by approximation $\int\int_{G/K\times G/K} h\varphi\,dx\,dy = 0$ for all continuous functions φ with compact support. But as h is locally integrable a standard result from real analysis implies $h = 0$ almost everywhere. $\square$

(It is also easy, and possibly more natural, to prove that $*$ is associative directly by a calculation.) This completes the proof of the theorem. $\square$

EXERCISE 3.2.8 (Relationship to Group Algebras). If the group G is unimodular, then the space $L^1(G)$ is a Banach algebra under the convolution product. (In the case of finite groups this is just the group algebra of G.) It this exercise we show that when $K = \{e\}$ is the trivial subgroup of G so that $G/K = G$ then is the same as the convolution algebra. We work with a space slightly different than $L^1(G)$ so that we can also deal with the case of non-unimodular functions.

For the rest of this exercise G is a Lie group and $K = \{e\}$ is the trivial subgroup of G. For any measurable function f on G define a function $K_f : G \times G \to \mathbf{R}$ by

$$K_f(x,y) = f(x^{-1}y). \tag{3.14}$$

(a) Show that the map $f \mapsto K_f$ is a bijection between the measurable functions on G and the set $\mathcal{M}(G;K)$.
For any measurable function f on G

$$\int_G |f(x^{-1})|^p\,dx = \int_G |f(x)|^p \Delta_G(x)\,dx.$$

Define a norm $\|\cdot\|_{L^p_\theta}$ for function on G by

$$\begin{aligned}
\|f\|_{L^p_\theta} &= \max\left\{\left(\int_G |f(x)|^p\,dx\right)^{\frac{1}{p}}, \left(\int_G |f(x^{-1})|^p\,dx\right)^{\frac{1}{p}}\right\} \\
&= \max\left\{\left(\int_G |f(x)|^p\,dx\right)^{\frac{1}{p}}, \left(\int_G |f(x)|^p\Delta_G(x)\,dx\right)^{\frac{1}{p}}\right\}.
\end{aligned}$$

When the group is unimodular this is just the usual L^p norm on G. Let $L^p_\theta(G)$ be the Banach space of measurable functions f on G with finite L^p_θ

norm. (Compare this to Proposition 3.3.1 and Theorem 3.3.2.)
(b) Show that the map $f \mapsto K_f$ is an isomorphism of the Banach spaces $L^p_\theta(G)$ and $L^p(G;K)$.
If $f_1, f_2 \in L^1_\theta(G)$ define the convolution product $f_1 \star f_2$ by

$$f_1 \star f_2(y) := \int_G f_1(z) f_2(z^{-1}y)\, dz. \tag{3.15}$$

(c) Show that for $f_1, f_2 \in L^1_\theta(G)$ that

$$K_{f_1} * K_{f_2} = K_{f_1 \star f_2}. \tag{3.16}$$

Thus the Banach algebras $(L^1(G;K), *)$ and $(L^1_\theta(G), \star)$ are isomorphic. In particular if $G = \mathbf{R}^n$ and $K = \{0\}$ then the the convolution algebra $L^1(\mathbf{R}^n;\{0\})$ defined above is naturally isomorphic to $L^1(\mathbf{R}^n)$ with the usual convolution product $f_1 \star f_2(y) = \int_{\mathbf{R}^n} f_1(y-z) f_2(z)\, dz$. □

3.3. Isotropic Functions and Approximations to the Identity

A function f defined on G/K is ***isotropic*** or ***radial*** iff $f(ax) = f(x)$ for all $a \in K$. If E is a G-module then denote by E^K the set of all isotropic functions in E, that is E^K is the set of elements of E invariant under K. Let $\mathcal{M}(G/K)^K$ be the set of measurable isotropic functions on G/K. Define the left and right restriction functions $\mathrm{Res}_L, \mathrm{Res}_R : \mathcal{M}(G;K) \to \mathcal{M}(G/K)^K$ by

$$(\mathrm{Res}_L\, h)(x) := h(x, \mathbf{o}), \quad (\mathrm{Res}_R\, h)(y) := h(\mathbf{o}, y)$$

Note if $a \in K$ then $a\mathbf{o} = \mathbf{o}$ so $(\mathrm{Res}_L\, h)(ax) = h(ax, \mathbf{o}) = h(ax, a\mathbf{o}) = h(x, \mathbf{o}) = (\mathrm{Res}_L\, h)(x)$ and thus $\mathrm{Res}_L\, h$ is isotropic. Likewise for $\mathrm{Res}_R\, h$. Define the left and right extension operators $\mathrm{Ext}_L, \mathrm{Ext}_R : \mathcal{M}(G/K)^K \to \mathcal{M}(G;K)$ by

$$\begin{array}{rcll} (\mathrm{Ext}_L\, f)(x,y) & := & f(\eta^{-1}x) & \text{where } \eta \in G \text{ with } \eta\mathbf{o} = y \\ (\mathrm{Ext}_R\, f)(x,y) & := & f(\xi^{-1}y) & \text{where } \xi \in G \text{ with } \xi\mathbf{o} = x \end{array}$$

If η' is anther element of G with $\eta'\mathbf{o} = y$ then $\eta' = \eta a$ for some $a \in K$ and as f is isotropic $f((\eta')^{-1}x) = f(a^{-1}\eta^{-1}x) = f(\eta^{-1}x)$ so that the definition of $\mathrm{Ext}_L\, f$ is independent of the choice of η with $\eta\mathbf{o} = y$. Likewise $\mathrm{Ext}_R\, f$ is independent of the choice of ξ with $\xi\mathbf{o} = x$. Also if $\eta\mathbf{o} = y$, then $g\eta\mathbf{o} = gy$ and so

$$(\mathrm{Ext}_L\, f)(gx, gy) = f((g\eta)^{-1}gx) = f(\eta^{-1}x) = (\mathrm{Ext}_L\, f)(x,y)$$

and thus $\mathrm{Ext}_L\, f \in \mathcal{M}(G;K)$. Similarly $\mathrm{Ext}_R\, f$ is in $\mathcal{M}(G;K)$. The maps Res_L and Ext_L are inverse of each other, and likewise for the right restriction and extension maps:

$$\mathrm{Res}_L \mathrm{Ext}_L\, f = f, \qquad \mathrm{Ext}_L \mathrm{Res}_L\, h = h \tag{3.17}$$

$$\mathrm{Res}_R \mathrm{Ext}_R\, f = f, \qquad \mathrm{Ext}_R \mathrm{Res}_R\, h = h \tag{3.18}$$

We check the first of these.

$$(\mathrm{Res}_L \mathrm{Ext}_L\, f)(y) = (\mathrm{Res}_L\, f)(x, \mathbf{o}) = f(x),$$

$$\begin{aligned}(\operatorname{Ext}_L \operatorname{Res}_L h)(x,y) &= (\operatorname{Res}_L h)(\eta^{-1}x) \qquad (\eta\mathbf{o} = y)\\ &= h(\eta^{-1}x,\mathbf{o})\\ &= h(x,\eta\mathbf{o})\\ &= h(x,y)\end{aligned}$$

It follows directly from the definitions for $f \in \mathcal{M}(G;K)$ that

$$\int_{G/K} |\operatorname{Ext}_L f(x,\mathbf{o})|^p\,dx = \int_{G/K} |f(x)|^p\,dx \tag{3.19}$$

$$\int_{G/K} |\operatorname{Ext}_R f(\mathbf{o},y)|^p\,dy = \int_{G/K} |f(y)|^p\,dy \tag{3.20}$$

For a function h to be in $L^p(G;K)$ both of the integrals $\int_{G/K} |h(x,\mathbf{o})|^p\,dx$ and $\int_{G/K} |h(\mathbf{o},y)|^p\,dy$ must be finite. To give the conditions on a function $f \in \mathcal{M}(G/K)^K$ so that $h = \operatorname{Ext}_L f$ satisfies these conditions we need a definition: Let $f \in \mathcal{M}(G/K)^K$, then define θf by

$$(\theta f)(x) := f(\xi^{-1}\mathbf{o}) \quad \text{(where } \xi\mathbf{o} = x.) \tag{3.21}$$

As f is isotropic this is independent of the choice of ξ with $\xi\mathbf{o} = x$. (For if $\xi'\mathbf{o} = x$ then $\xi' = \xi a$ for some $a \in K$ and $f((\xi')^{-1}\mathbf{o}) = f(a^{-1}\xi^{-1}\mathbf{o}) = f(\xi^{-1}\mathbf{o})$.) To give a different interpitation of θ if $h(x,y) = (\operatorname{Ext}_R f)(x,y) = f(\xi^{-1}y)$ where $\xi\mathbf{o} = x$ then $(\operatorname{Res}_L h)(x) = h(x,\mathbf{o}) = f(\xi^{-1}\mathbf{o}) = (\theta f)(x)$. That is

$$\theta f = \operatorname{Res}_L \operatorname{Ext}_R f \tag{3.22}$$

Let $L^p_\theta(G/K)^K$ be the set of measurable isotropic functions f so that the norm

$$\|f\|_{\theta,p} := \max\left\{ \left(\int_{G/K} |f(x)|^p\,dx\right)^{\frac{1}{p}}, \left(\int_{G/K} |(\theta f)(y)|^p\,dy\right)^{\frac{1}{p}} \right\} \tag{3.23}$$

is finite.

Proposition 3.3.1. *The map* $\operatorname{Ext}_R : L^p_\theta(G/K)^K \to L^p(G;K)$ *is an bijective isometry of Banach spaces.*

Proof. If $f \in \mathcal{M}(G/K)^K$ then by the formulas above

$$\int_{G/K} |(\operatorname{Ext}_R f)(\mathbf{o},y)|^p\,dy = \int_{G/K} |f(y)|^p\,dy$$

and

$$\begin{aligned}\int_{G/K} |(\operatorname{Ext}_R f)(x,\mathbf{o})|^p\,dx &= \int_{G/K} |(\operatorname{Res}_L \operatorname{Ext}_R f)(x)|^p\,dx\\ &= \int_{G/K} |(\theta f)(x)|^p\,dx.\end{aligned}$$

Therefore the result follows form the definitions of the norms on $L^p(G;K)$ and $L^p_\theta(G/K)^K$. $\square$

When the group G is unimodular this simplifies:

THEOREM 3.3.2. *If the group G is unimodular then for all $h \in \mathcal{M}(G;K)$*

$$\int_{G/K} |h(x,\mathbf{o})|^p\,dx = \int_{G/K} |h(\mathbf{o},y)|^p\,dy \tag{3.24}$$

PROOF. From Proposition 3.3.1 and its proof it follows it is enough to prove for unimodular G that

$$\int_{G/K} |(\theta f)(x)|^p\,dx = \int_{G/K} |f(x)|^p\,dx$$

for all $f \in \mathcal{M}(G/K)^K$. If $f \in \mathcal{M}(G/K)^K$ define a measurable function $f^\#$ on G by $f^\# := \pi^* f = f \circ \pi$ where $\pi : G \to G/K$ is the natural projection. If $\pi\xi = x$ then $\xi\mathbf{o} = x$ and $(\theta f)(x) = f(\xi^{-1}\mathbf{o})$. Thus

$$(\theta f)^\#(\xi) = (\theta f)(x) = f(\xi^{-1}\mathbf{o}) = f(\pi\xi^{-1}) = f^\#(\xi^{-1})$$

We can assume that the left invariant measure is the Riemannian measure of a left invariant Riemannian metric on G. Then for any function f on G/K

$$\mathrm{Vol}(K) \int_{G/K} f(x)\,dx = \int_G f^\#(\xi)\,d\xi.$$

Also if Δ_G is the modular function of G then under the map $\xi \mapsto \xi^{-1}$ the invariant measure $d\xi$ maps by $d\xi \mapsto \Delta_G(\xi)\,d\xi$. Putting these facts together for any $f \in \mathcal{M}(G/K)^K$

$$\mathrm{Vol}(K) \int_{G/K} |(\theta f)(x)|^p\,dx = \int_G |f^\#(\xi^{-1})|^p\,d\xi = \int_G |f^\#(\xi)|^p \Delta_G(\xi)\,d\xi$$

$$\mathrm{Vol}(K) \int_{G/K} |f(x)|^p\,dx = \int_G |f^\#(\xi)|^p\,d\xi.$$

So equation (3.24) holds if and only if

$$\int_G |f^\#(\xi)|^p \Delta_G(\xi)\,d\xi = \int_G |f^\#(\xi)|^p\,d\xi. \tag{3.25}$$

If G is unimodular then $\Delta_G \equiv 1$ and this certainly holds. $\square$

EXERCISE 3.3.3. Use equation (3.25) to show that the condition (3.24) holds if and only if G is unimodular. $\square$

EXERCISE 3.3.4. Assume that G is unimodular let $1 \le p < \infty$. Set $p' = p/(p-1)$. Then show the dual space (i.e. the space of continuous linear functionals) of $L^p(G;K)$ is $L^{p'}(G;K)$ and the pairing between the spaces is

$$\langle h,k\rangle := \int_{G/K} h(x,\mathbf{o})k(x,\mathbf{o})\,dx \qquad \text{for } h \in L^p(G;K) \text{ and } k \in L^{p'}(G;K) \quad \square$$

Next we construct invariant smoothing operators. As the group K is compact the homogeneous space G/K will have a left invariant Riemannian metric $\langle\,,\rangle$ (Proposition 2.3.12). Let $d(x,y)$ be the Riemannian distance between x and y defined by the metric $\langle\,,\rangle$. Then the function $(x,y)\mapsto d(x,y)^2$ on $G/K\times G/K$ is smooth in a neighbor hood of the diagonal $\{x=y\}$. Let $\varphi:\mathbf{R}\to[0,\infty)$ so that $\varphi(-t)=\varphi(t)$ and the support of φ is contained in $[-1,1]$. Then for $\delta>0$ define $\Phi_\delta: G/K\times G/K$ by

$$\Phi_\delta(x,y) := C(\delta)\varphi\left(\frac{d(x,y)^2}{\delta^2}\right)$$

where $C(\delta)\int_{G/K}\varphi\left(d(x,\mathbf{o})^2/\delta^2\right)\,dx = 1$. This satisfies

$$\begin{aligned}
\Phi_\delta(gx,gy) &= \Phi_\delta(x,y) && \text{for all } g\in G\\
\Phi_\delta(y,x) &= \Phi_\delta(x,y) &&\\
\int_{G/K}\Phi_\delta(x,y)\,dy &= 1 && \text{for all } x\in G/K\\
\Phi_\delta(x,y) &= 0 && \text{if } d(x,y)\ge\delta\\
\Phi_\delta &\in C^\infty(G/K\times G/K) && \text{for small } \delta\\
\Phi_\delta(x,y) &\ge 0. &&
\end{aligned}$$

Recall $L^p_{\mathrm{Loc}}(G/K)$ is the set of all measurable functions f on G/K with $\int_C |f(x)|^p\,dx<\infty$ for all compact subsets C of G/K.

Theorem 3.3.5. *Let $f\in L^p_{\mathrm{Loc}}(G/K)$ and define f_δ by*

$$f_\delta(x) := T_{\Phi_\delta}f(x) = \int_{G/K}\Phi_\delta(x,y)f(y)\,dy. \tag{3.26}$$

Then for all small δ the function f_δ is in $C^\infty(G/K)$. Also $\lim_{\delta\searrow 0} f_\delta(x) = f(x)$ *for almost all $x\in G/K$. If $1\le p<\infty$ and $f\in L^p(G/K)$, then* $\lim_{\delta\searrow 0}\|f-f_\delta\|_{L^p}=0$. *If $f\in C^k(G/K)$ for some $k\ge 0$ then $f_\delta\to f$ in the C^k topology uniformly on compact subsets of G/K.*

Proof. An exercise based on the above properties of Φ_δ. □

3.4. Symmetric and Weakly Symmetric Spaces

Let G/K be a homogeneous space with K compact and let $\mathbf{o}=\pi(e)$ be the origin as usual. Then G/K is a ***symmetric space*** iff there is an element $\iota_{\mathbf{o}}\in K$ so that the derivative $\iota_{\mathbf{o}*}$ of $\iota_{\mathbf{o}}$ at $\mathbf{o}$ satisfies

$$\iota_{\mathbf{o}*} = -\operatorname{Id}|_{T(G/K)_{\mathbf{o}}} \tag{3.27}$$

If G/K is a symmetric space and $x\in G/K$ then choose $\xi\in G$ with $\xi\mathbf{o}=x$, the ***symmetry at*** x is defined by

$$\iota_x = \xi\iota_{\mathbf{o}}\xi^{-1}. \tag{3.28}$$

Then the derivative of ι_x at x is

$$\iota_{x*} = -\operatorname{Id}|_{T(G/K)_x}. \tag{3.29}$$

As the group K is compact we can assume that G/K has in invariant Riemannian metric $\langle\,,\rangle$ (cf. 2.3.12). Let $\exp_x : T(G/K)_x \to G/K$ be the ***exponential of this Riemannian metric***. (That is for each $X \in T(G/K)_x$ the map $\gamma(x) = \exp_x(tX)$ is the geodesic so that $\gamma(0) = x$ and $\gamma'(0) = X$.) As ι_x is a isometry

$$\iota_x(\exp_x(X)) = \exp_x(-X). \tag{3.30}$$

For this reason ι_x is often called the ***geodesic symmetric*** at x.

PROPOSITION 3.4.1 (Gel'fand). *If G/K is a symmetric space then every $h \in \mathcal{M}(G;K)$ is symmetric: $h(x,y) = h(y,x)$.*

PROOF. Let $x, y \in G/K$ and let $2\ell = d(x,y)$ be the Riemannian distance between x and y. Then there is a minimizing unit speed geodesic $\gamma : [-\ell, \ell]$ form x to y with $\gamma(-\ell) = x$ and $\gamma(\ell) = y$. Let $z = \gamma(0)$ be the midpoint of this segment. Then the geodesic symmetry ι_z satisfies $\iota_z(x) = \iota_z(\gamma(-\ell)) = \gamma(\ell) = y$, and likewise $\iota_z(y) = x$. Thus form the symmetry condition defining ι_z

$$h(x,y) = h(\iota_z(x), \iota_z(y)) = h(y,x).$$

□

In terms of the harmonic analysis on G/K the symmetry of the functions $h \in \mathcal{M}(G;K)$ is almost as important as the existence of the geodesic symmetries. So we define a homogeneous space G/K with K compact to be ***weakly symmetric*** iff

$$h \in \mathcal{M}(G;K) \quad \text{implies} \quad h(x,y) = h(y,x) \quad \text{for all } x, y \in G/K. \tag{3.31}$$

As examples of weakly symmetric spaces consider the sphere S^n as homogeneous spaces $S^n = SO(n+1)/SO(n)$. If $e_1, \dots, e_{n+1}$ is an orthonormal basis of $\mathbf{R}^n$ then the symmetry at e_1 has the matrix representation

$$\iota_{e_1} = \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix}$$

where I is the $n \times n$ identity matrix. This is in $SO(n+1)$ if and only if $(-1)^n = 1$, that is if and only if n is even. However it is an easy exercise to show that as a homogeneous space $S^n = SO(n+1)/SO(n)$ is a weakly symmetric space.

EXERCISE 3.4.2. Show that $S^n = SO(n+1)/SO(n)$ is a weakly symmetry space. HINT: Show that if $x, y \in S^n$ there is a $g \in SO(n+1)$ with $gx = y$ and $gy = x$ and then argue as in the proposition. □

THEOREM 3.4.3. *If G/K is a weakly symmetric space then the group G is unimodular and the norms on the spaces $L^p(G;K)$ are given by*

$$\|h\|_p = \left(\int_{G/K} |h(x,\mathbf{o})|^p\,dx\right)^{\frac{1}{p}} = \left(\int_{G/K} |h(\mathbf{o},y)|^p\,dy\right)^{\frac{1}{p}}. \tag{3.32}$$

Also the convolution algebra is commutative. That is for all $h, k \in L^1(G;K)$ $h * k = k * h$.

REMARK 3.4.4. This original version of this result is due to Gel'fand [**13**].

PROOF. The symmetry property of $h \in \mathcal{M}(G;K)$ implies

$$\int_{G/K} |h(x, \mathbf{o})|\, dx = \int_{G/K} |h(\mathbf{o}, x)|\, dx.$$

By Exercise 3.3.3 this implies G is unimodular. That the norm on $L^p(G;K)$ is given by (3.32) follows directly form the symmetry of the functions h. Finally for $h, k \in L^1(G;K)$ using the symmetry of h, k and $k * h$

$$\begin{aligned}(h * k)(x, y) &= \int_{G/K} h(x, z)k(z, y)\, dz \\ &= \int_{G/K} k(y, z)h(z, x)\, dz \\ &= (k * h)(y, x) \\ &= (k * h)(x, y).\end{aligned}$$

and $L^1(G;K)$ is commutative as claimed. □

In the case G/K is a weakly symmetric space the relationship between $L^p(G;K)$ and $L^p(G/K)^K$ given in section 3.3 shows there is no need to distinguish the left and right restrictions or between the left and right extensions. For for future use we record:

PROPOSITION 3.4.5. *Let* G/K *be a weakly symmetric space. Then for* $1 \le p \le \infty$ *there there are Banach space isomorphisms* $\operatorname{Res} : L^p(G;K) \to L^p(G/K)^K$ *given by*

$$(\operatorname{Res} h)(x) := h(x, \mathbf{o}) = h(\mathbf{o}, x).$$

This has as inverse $\operatorname{Ext} : L^p(G/K)^K \to L^p(G;K)$ *given by*

$$\begin{aligned}(\operatorname{Ext} f)(x, y) &= f(\xi^{-1} y) \qquad (\textit{where } \xi\mathbf{o} = x) \\ &= f(\eta^{-1} x) \qquad (\textit{where } \eta\mathbf{o} = y).\end{aligned}$$

PROOF. This follows form the results of section 3.3 and the symmetry property $h(x, y) = h(y, x)$. □

CHAPTER 4

Compact Groups and Homogeneous Spaces

4.1. Complete Reducibility of Representations

Our goal in this section is to show that many representations of compact groups can be decomposed into direct sums of finite dimensional irreducible representations. The basic method is to construct (by averaging) an invariant inner product on the G-module in question and then showing that the orthogonal complement of a submodule is also a submodule.

Recall form section 3.1.1 that a representation $\rho : G \to GL(X)$ of a Banach space X is strongly continuous iff the map $\xi \mapsto \rho(\xi)x$ is norm continuous for each $x \in X$.

Let X and Y be a Banach space then the ***operator norm*** of a linear operator $A : X \to Y$ is

$$\|A\|_{\mathrm{Op}} := \sup_{0 \neq x \in X} \frac{\|Ax\|_Y}{\|x\|_X}.$$

The operator norm defines a norm on the vector space of bounded linear maps form X to Y.

PROPOSITION 4.1.1. *Let G be a compact group and $\mathcal{H}$ a Hilbert space with inner product $(\,,)$. Assume that $\rho :\to GL(\mathcal{H})$ is strongly continuous representation of G on $\mathcal{H}$. Then there is inner product $\langle\,,\rangle$ on $\mathcal{H}$ which is invariant under G (i.e. $\langle\rho(g)x, \rho(g)y\rangle = \langle x, y\rangle$ and which is equivalent to $(\,,)$ in the sense that there is a constant c so that $c^{-1}(x, x) \leq \langle x, x\rangle \leq c(x, x)$.*

PROOF. As G is compact there is an bi-invariant measure $d\xi$ on G which we can assume to have total mass 1. Define $\langle\,,\rangle$ by

$$\langle x, y\rangle = \int_G (\rho(\xi)x, \rho(\xi)y)\, d\xi.$$

That $\langle\,,\rangle$ is an inner product is easy to check. Using a change of variable $\xi \mapsto g^{-1}\xi$

$$\langle\rho(g)x, \rho(g)y\rangle = \int_G (\rho(g\xi), x, \rho(g\xi)y)\, d\xi = \int_G (\rho(\xi)x, \rho(\xi)y)\, d\xi = \langle x, y\rangle.$$

Thus $\langle\,,\rangle$ is invariant. If $x \in \mathcal{H}$ the $\xi \mapsto \|\rho(\xi)x\|_{\mathcal{H}}$ is continuous and G is compact so $\sup_{\xi\in G} \|\rho(\xi)x\|_{\mathcal{H}} < \infty$. As this holds for all $x \in X$ the uniform boundedness principle (cf. Theorem A.3.1 in the appendix) implies there is a constant C so that $\|\rho(\xi)\|_{\mathrm{Op}} \le C$ for all $\xi \in G$. This implies $(\rho(\xi)x, \rho(\xi)x) \le C^2(x,x)$. Thus

$$\langle x, x\rangle = \int_G (\rho(\xi)x, \rho(\xi))\, d\xi \le C^2 \int_G (x,x)\, d\xi = C^2(x,x).$$

Also

$$(x,x) = (\rho(\xi^{-1})\rho(\xi)x, \rho(\xi^{-1})\rho(\xi)x) \le C^2(\rho(\xi)x, \rho(\xi)x)$$

so that $(\rho(\xi)x, \rho(\xi)x) \ge C^{-2}(x,x)$. Similar calculation show $C^{-2}(x,x) \le \langle x, x\rangle$. ☐

EXERCISE 4.1.2. As a generalization of this show that if $\rho : G \to GL(X)$ is a strongly continuous representation on the Banach space X with norm $\|\cdot\|_X$, then X has a new norm $|\ |_X$ that is invariant under G (i.e. $|\rho(g)x|_X = |x|_X$) and so that for some $C > 0$ $C^{-1}\|x\|_X \le |x|_X \le C\|x\|_X$ for all $x \in X$. HINT: Define $|x|_X = \int_G \|\rho(\xi)x\|_X\, d\xi$. ☐

PROPOSITION 4.1.3. *Let G be a compact group, $\mathcal{H}$ a Hilbert space, and $\rho : G \to GL(\mathcal{H})$ and assume that the inner product $\langle\, ,\rangle$ of $\mathcal{H}$ is invariant under G. If E is a G-submodule of $\mathcal{H}$, then so is the orthogonal complement $E^\perp$ of E.*

PROOF. Let $x \in E^\perp$ and $g \in G$. Then for any $y \in E$,

$$\langle \rho(g)x, y\rangle = \langle \rho(g^{-1})\rho(g)x, \rho(g^{-1})y\rangle = \langle x, \rho(g^{-1})y\rangle = 0$$

as $\rho(g^{-1})y \in E$ and $x \in E^\perp$. Thus $\rho(g)x \in E^\perp$. ☐

COROLLARY 4.1.4. *If G is compact, E is a finite dimensional G-module, and E_1 is a G-submodule of E, then there is a G-submodule E_2 so that $E = E_1 \oplus E_2$.*

PROOF. As E is finite dimensional there is at least one inner product on E. By proposition 4.1.1 we can assume that E this inner product is invariant under G. Let E_2 be the the orthogonal complement of E_1 with respect to this inner product. Then $E = E_1 \oplus E_2$ and by the last proposition E_2 is a G-submodule. ☐

The following result, due to Hermann Weyl, is basic to the theory of compact groups.

THEOREM 4.1.5 (Weyl). *If G is a compact group and E is a finite dimensional G-module then E is a direct sum $E = E_1 \oplus \cdots \oplus E_n$ of irreducible G-submodules.*

PROOF. Let E_1 be a G-submodule of E of minimal dimension. Then E_1 is irreducible. By the last corollary $E = E_1 \oplus F$ for some G-submodule F. The result now follows by induction on $\dim E$. ☐

If the group is not compact then this result need not be true. As an example let G be the additive group of the real numbers $(\mathbf{R}, +)$ and let ρ be the representation on $\mathbf{R}^2$ given by $\rho(t) := \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. The only submodule of $\mathbf{R}^2$ is $E_1 := \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbf{R} \right\}$. Thus $\mathbf{R}^2$ is not a direct sum of irreducibles and the submodule E_1 has no complementary submodule. If is also not hard to see that there is no inner product on $\mathbf{R}^2$ that is invariant under ρ. On the other hand if the group is semisimple, then any finite dimensional G-module is a direct sum of irreducible G-submodules and every submodule has a complementary submodule. This result (also due to Hermann Weyl) is deeper than the results above about compact groups. (Although Weyl's original argument reduces the semisimple case to the compact case by showing that every semisimple group G contains a compact group K that is dense in the Zarski topology. It follows form this that in any finite dimensional representation of G on a finite dimensional space E that a subspace V is a G submodule if and only if it is a K-submodule. However proving the existence of the compact subgroup K requires a fair amount of work.)

Exercise 4.1.6. This is for readers who know a little of the theory of several complex variables and which want a concrete example of the remarks of the last paragraph. Let $G = GL(n, \mathbf{C})$ be the group of complex $n \times n$ matrices. Then G is a complex analytic manifold in a natural way. Let $K = U(n)$ be the subgroup of unitary matrices in G.
(a) Show that any holomorphic (i.e. complex analytic) function on G that vanishes on K also varnishes on G. Thus two holomorphic functions that agree on K are equal.
(b) Let E be a finite dimensional complex vector space. Call a representation $\rho : G \to GL(E)$ holomorphic iff the component functions of the matrices representing ρ are holomorphic. If ρ is a holomorphic representation show that a subspace V of E is a G submodule if and only if it is a K submodule. Hint: Let V be a K-submodule of E, and $\ell : E \to \mathbf{C}$ be a linear function that vanishes on V. Then for any $v \in V$ the function $\xi \mapsto \ell(\rho(\xi))$ is a holomorphic function on G that vanishes on K and this it also vanishes on G. But v was any element of V and ℓ any element linear functional vanishing on V.
(c) Show that any holomorphic representation $\rho : G \to GL(E)$ of G is a direct sum of irreducible representations. Hint: Decompose E under the action of K and use part (b).
(d) Consider the representation $\rho : G \to GL(\mathbf{C}^2)$ given by

$$\rho(g) = \begin{bmatrix} 1 & \log|\det(g)| \\ 0 & 1 \end{bmatrix}.$$

Then this representation is not a direct sum of irreducible representations. (But it is not holomorphic. Also the group $GL(n, \mathbf{C})$ is not semisimple. The group $SL(n, \mathbf{C})$ is semisimple and thus all of its representations are a direct sum of irreducibles.) $\square$

Our next goal is to extend theorem 4.1.5 to infinite dimensional Hilbert spaces. The hard part of the proof is to show that a representation on a Hilbert space must have a finite dimensional submodule:

LEMMA 4.1.7. *Let $\rho : G \to GL(\mathcal{H})$ be a strongly continuous representation of the compact group G on the Hilbert space $\mathcal{H}$ and assume that the inner product $\langle\, ,\rangle$ is invariant under the action of G. Then $\mathcal{H}$ has a finite dimensional irreducible submodule.*

PROOF. The idea is to find a compact self-adjoint linear A map on $\mathcal{H}$ that commutes with the action of G and then to find the required submodule as a submodule of one of the eigenspaces of $\mathcal{H}$. Let $0 \neq v \in V$. We can assume that $\|v\|_{\mathcal{H}} = 1$. We normalized the invariant measure $d\xi$ to have total mass 1. Define $A : \mathcal{H} \to \mathcal{H}$ by

$$Ax := \int_G \langle x, \rho(\xi)v\rangle \, d\xi$$

As the inner product is preserved by G, $\|\rho(\xi)x\|_{\mathcal{H}} = \|x\|_{\mathcal{H}}$ for all $\xi \in G$. As v has length one it follows $\|Ax\|_{\mathcal{H}} \leq \int_G ||\langle x, \rho(\xi)v\rangle| \, d\xi \leq \|x\|_{\mathcal{H}}$. Thus A is bounded with operator norm $\|A\|_{\mathrm{Op}} \leq 1$.

$$\begin{aligned}
A\rho(g)x &= \int_G \langle \rho(g)x, \rho(\xi)v\rangle \rho(\xi)v \, d\xi \\
&= \int_G \langle \rho(g\xi), \rho(g\xi)v\rangle \rho(g\xi)v \, d\xi \qquad \text{(change of variable } \xi \mapsto g\xi) \\
&= \int_G \langle x, \rho(\xi)v\rangle \rho(g)\rho(\xi)v \, d\xi \\
&= \rho(g)Ax.
\end{aligned}$$

Thus A is an intertwining map.

$$\begin{aligned}
\langle Ax, y\rangle &= \left\langle \int_G \langle x, \rho(\xi)v\rangle \rho(\xi)v \, d\xi, y \right\rangle \\
&= \int_G \langle x, \rho(\xi)v\rangle \langle \rho(\xi)v, y\rangle \, d\xi \\
&= \left\langle \int_G \langle x, \rho(\xi)v\rangle \rho(\xi)v \, d\xi \right\rangle \\
&= \langle x, Ay\rangle
\end{aligned}$$

which shows A is self-adjoint.

$$\begin{aligned}\langle Av, v\rangle &= \left\langle \int_G \langle v, \rho(\xi)v\rangle \rho(\xi)v \, d\xi, v \right\rangle \\ &= \int_G \langle v, \rho(\xi)v\rangle \langle \rho(\xi)v, v\rangle \, d\xi \\ &= \int_G |\langle v, \rho(\xi)v\rangle|^2 \, d\xi > 0\end{aligned}$$

Which shows $A \neq 0$.

We now claim A is compact. Let $\|\cdot\|_{\mathrm{Op}}$ be the operator norm. It is a basic result (cf. A.3.2) that if a linear operator can be approximated in the operator norm by finite rank operators then it is compact. As the group G is compact there is a bi-invariant Riemannian metric on it. Let $d : G \times G \to [0, \infty)$ be the distance function of the invariant Riemannian. Let $\varepsilon > 0$. Then as G is compact and $\rho : G \to GL(\mathcal{H})$ is strongly continuous there is a $\delta = \delta_\varepsilon$ so that if $\xi, \eta \in G$ and $d(\xi, \eta) < \delta$, then $\|\rho(\xi)v - \rho(\eta)v\|_{\mathcal{H}} < \varepsilon$. Again using that G is compact there is a finite open cover $\{U_1, \dots, U_m\}$ of G so that if $\xi, \eta \in U_i$, then $d(\xi, \eta) < \delta$ and thus if $\xi, \eta \in U_i$ then $\|(\rho(\xi) - \rho(\eta))v\|_{\mathcal{H}} < \varepsilon$. For each i choose $\xi_i \in U_i$. Let $\{\varphi_i\}_{i=1}^m$ be a partition of unity subordinate to the cover $\{U_1\}_{i=1}^m$, that is each φ_i is continuous and non-negative, the support of φ_i is contained in U_i and $\sum_{i=1}^m \varphi = 1$. Define a linear operator A_i by

$$A_i x := \int_G \langle x, \rho(\xi)v\rangle \varphi_i(\xi)\rho(\xi)v \, d\xi$$

As the φ_i's sum to 1

$$A = \sum_{i=1}^m A_i.$$

Define a rank one operator B_i (with range spanned by $\rho(\xi_i)v$) and a finite rank operator B by

$$B_i x := \int_G \langle x, \rho(\xi)v\rangle \varphi_i(\xi)\rho(\xi_i)v \, d\xi = \int_G \langle x, \rho(\xi)v\rangle \varphi_i(\xi) \, d\xi \, \rho(\xi_i)v$$

$$B = \sum_{i=1}^m B_i.$$

If ξ is in the support of φ_i then both ξ and ξ_i and in U_i and thus $\|\rho(\xi)v - \rho(\xi_i)v\|_{\mathcal{H}} < \varepsilon$. As $\varphi_i(\xi)$ vanishes for all ξ not in the support of φ_i and $\|\rho(\xi)v\|_{\mathcal{H}} = 1$

$$\begin{aligned}&\|\langle x, \rho(\xi)v\rangle \varphi_i(\xi)\rho(\xi)v - \langle x, \rho(\xi)v\rangle \rho(\xi)v\|_{\mathcal{H}}\varphi_i(\xi) \\ &\qquad = |\langle x, \rho(\xi)v\rangle| \varphi_i(\xi) \|\rho(\xi)v - \rho(\xi_i)v\|_{\mathcal{H}} \leq \varepsilon \|x\|_{\mathcal{H}} \varphi_i(\xi)\end{aligned}$$

and thus

$$\begin{aligned}\|A_i x - B_i x\|_{\mathcal{H}} &\leq \int_G \|\langle x, \rho(\xi)v\rangle \varphi_i(\xi)\rho(\xi)v - \langle x, \rho(\xi)v\rangle\varphi_i(\xi)\rho(\xi_i)v\|_{\mathcal{H}}\, d\xi \\ &\leq \varepsilon\|x\|_{\mathcal{H}} \int_G \varphi_i(\xi)\, d\xi\end{aligned}$$

and (using $\int_G 1\, d\xi = 1$)

$$\|Ax - Bx\|_{\mathcal{H}} \leq \sum_{i=1}^m \|A_i x - B_i x\|_{\mathcal{H}} \leq \varepsilon\|x\|_{\mathcal{H}} \sum_{i=1}^m \int_G \varphi_i(\xi)\, d\xi = \varepsilon\|x\|_{\mathcal{H}}$$

which implies $\|A - B\|_{\mathrm{Op}} < \varepsilon$. As ε was arbitrarily this show A can be norm approximated by finite rank operators and competes the proof that A is compact.

A nonzero compact self-adjoint linear operator A has at least one nonzero eigenvalue α. Let $E_\alpha = \{x \in \mathcal{H} : Ax = \alpha x\}$ be the corresponding eigenspace. If $x \in E_\alpha$ then $A\rho(g)x = \rho(g)Ax = \alpha\rho(g)x$ and thus E_α is a G-submodule of $\mathcal{H}$. As A is compact and $\alpha \neq 0$ the space E_α is finite dimensional (this follows from Theorem A.2.1 where $\mathcal{A}$ is just taken to be the set of scalar multiplies of A). Let E be a G-submodule of E_α of minimal dimension. Then E will be a finite dimensional irreducible G-submodule of $\mathcal{H}$. □

Theorem 4.1.8. *Let $\rho : G \to GL(\mathcal{H})$ be a strongly continuous representation of the compact group G on the Hilbert space $\mathcal{H}$ and assume that the inner product $\langle\,,\rangle$ is invariant under the action of G. Then $\mathcal{H}$ is an orthogonal direct sum*

$$\mathcal{H} = \bigoplus_\alpha E_\alpha$$

of finite dimensional irreducible G-modules E_α.

Remark 4.1.9. I am not sure of the history of this result. When $\mathcal{H}$ is finite dimensional it is due to Weyl and it is likely that Weyl also know the infinite dimensional. There is a more general version for representations of compact groups on locally convex topological vector spaces. This can be found in Helgason [**17**, Thm 1.6 p. 392] with some of the history to be found in the notes [**17**, pp. 491–492].

Proof. Let $\mathcal{B}$ be the collection of all subsets $\mathcal{E} = \{E_\alpha\}$ where each E_α is a finite dimensional irreducible G-submodule of $\mathcal{H}$ and so that if $E_\alpha, E_\beta \in \mathcal{E}$, with $E_\alpha \neq E_\beta$ then $E_\alpha \perp E_\beta$. By the lemma there a finite dimensional irreducible G submodule E of $\mathcal{H}$ and thus $\mathcal{E} = \{E\} \in \mathcal{B}$. So $\mathcal{B}$ is not empty. Order $\mathcal{B}$ by inclusion and let $\mathcal{C}$ be a chain in $\mathcal{B}$. Then the union $\bigcup \mathcal{C}$ is in $\mathcal{B}$ and thus every chain has an upper bound. Therefore by Zorn's lemma $\mathcal{B}$ has a maximal element $\mathcal{E}_0 = \{E_\alpha : \alpha \in A\}$. Let $E = \bigoplus_{\alpha\in A} E_\alpha$. If $E \neq \mathcal{H}$ then $E^\perp \neq \{0\}$ and by proposition 4.1.3 $E^\perp$ is a submodule of $\mathcal{H}$. By the last lemma $E^\perp$ will have a finite dimensional irreducible submodule E'. But then $\mathcal{E}' := \{E'\} \cup \mathcal{E}_0 \in \mathcal{B}$, which contradicts the maximality of $\mathcal{E}_0$. Thus $E = \bigoplus_{\alpha \in A} E_\alpha = \mathcal{H}$. $\square$

4.1.1. Decomposition of $L^2(G)$ and $L^2(G/K)$. Let G be a compact Lie group and K a close subgroup. We can apply the results above to the special case of the regular representation τ of G on $L^2(G/K)$.

THEOREM 4.1.10. *Let G be a compact Lie group and K a closed subgroup of G. Then there is an orthogonal direct sum*

$$L^2(G/K) = \bigoplus_{\alpha \in A} E_\alpha$$

where each E_α is a finite dimensional irreducible G-submodule (under the regular representation $\tau_g f(x) = f(g^{-1}x)$).

PROOF. By proposition 3.1.7 and theorem 3.1.10 and the representation is strongly continuous and preserves the inner product. Thus this is a special case of theorem 4.1.8. $\square$

Let G be a compact Lie group and let $\tau : G \to GL(L^2(G))$ be the (left) regular representation $\tau_g f(\xi) = f(g^{-1}\xi)$ of G on $L^2(G)$. The following theorem shows that in at least one sense all the information about finite dimensional representations of G is contained in the regular representation.

THEOREM 4.1.11. *Let G be a compact group and $\rho : G \to GL(V)$ a finite dimensional representation of G. Then $L^2(G)$ contains a G-submodule E isomorphic to V.*

PROOF. By averaging we can assume V has in inner product invariant under G. Fix any non-zero vector $v_0 \in V$ and define a function $\varphi : E \to L^2(G)$ by

$$\varphi(v)(\xi) = \langle v, \rho(\xi)v_0 \rangle. \tag{4.1}$$

Clearly $v \mapsto \varphi(v)$ is linear and

$$\varphi(\rho(g))(\xi) = \langle \rho(g), \rho(\xi)v_0 \rangle = \langle v, \rho(g^{-1}v_0) = (\tau_g \varphi(v))(\xi).$$

Therefore $\varphi \circ \rho(g) = \tau_g \circ \varphi$ and thus φ is an intertwining map. As $\varphi(v_0)(e) = \langle v_0, v_0 \rangle \neq 0$ the map φ is not the zero map. Whence by Schur's lemma the $\varphi : V \to \varphi[V]$ is an isomorphism. Thus $E := \varphi[V]$ is the required G-submodule of $L^2(G)$. $\square$

Recall that if V is a G-module, and K is a subgroup of G then $V^K := \{v \in V : av = v\}$ is the subspace of all vector invariant by K.

THEOREM 4.1.12. *Let G be a compact group and K a closed subgroup of G. Let V be an irreducible finite dimensional G-module. Then $L^2(G/K)$ has a submodule isomorphic to V if and only if $V^K \neq \{0\}$.*

EXERCISE 4.1.13. Prove the last theorem. HINT: if $V^K \neq \{0\}$ then we can assume that V has an invariant inner product and choose $0 \neq v_0 \in V^K$. Define $\varphi : V \to L^2(G/K)$ by equation (4.1) and use that $v_0 \in V^K$ to show this is well defined and an intertwining map. Then Schur's lemma shows $E = \varphi[V]$ is a G-submodule of $L^2(G/K)$ isomorphic to V.

For the converse assume that G/K has a invariant Riemannian metric (which exists by 2.3.12) and let $d(x, y)$ be the Riemannian metric between $x, y \in G/K$. For $x \in G/K$ and $r > 0$ let $B(x, r) := \{y \in G/K : d(x, y) < r\}$ be the ball of radius r about x. For any function $f \in L^2(G/K)$ if $\int_{B(x_0,r)} f(x)\,dx = 0$ for all $x_0 \in G/K$ and $r > 0$ then $f = 0$ almost everywhere. Therefore if $V \subset L^2(G/K)$ is a non-zero irreducible submodule of there is $f_0 \in V$ and a ball $B(x_0, r)$ so that $\int_{B(x_0,r)} f_0(x)\,dx \neq 0$. As V is invariant under G we can assume that $x_0 = \mathbf{o}$ so that $\int_{B(\mathbf{o},r)} f_0(x)\,dx \neq 0$ for some $f_0 \in V$. Define a linear functional $\Lambda : V \to \mathbf{R}$ by $\Lambda(f) = \int_{B(\mathbf{o},r)} f(x)\,dx$. As the metric $d(\cdot, \cdot)$ is invariant under that action of G we have $aB(\mathbf{o}, r) = aB(\mathbf{o}, r)$ for all $a \in K$ which implies $\Lambda(\tau_a f) = \Lambda(f)$ for all $a \in K$. Finally we can represent Λ as an inner product, that there is an $h \in V$ for that $\Lambda(f) = \int_{G/K} f(x)h(x)\,dx$ for all $f \in V$ (as V is finite dimensional this only requires linear algebra). Now check $0 \neq h \in V^K$. □

4.1.2. Characters of Compact Groups. We now show that for compact groups that finite dimensional representations are determined by their characters.

If V is a finite dimensional complex vector space with a Hermitian inner product $\langle\,,\rangle$ then $U(V)$ will denote the unitary group of V. Let G be a compact Lie group. Then in this section we will only consider finite dimensional unitary representations of G. (Note by averaging (cf. Prop. 4.1.1) any finite dimensional representation is equivalent to a unitary representation so this is not a restriction.)

THEOREM 4.1.14. *If two finite dimensional representations ρ_1 and ρ_2 of the compact Lie group G have the same character they are equivalent. (Note that we are not assuming that the representations are irreducible.)*

LEMMA 4.1.15. *Let $\rho_1 : G \to U(V)$ and $\rho_2 : G \to U(W)$ be two irreducible representations of G and let $B : U \times W \to \mathbf{C}$ be a linear with respect to the first variable and conjugate linear with respect to the second slot. (I.e. $B(cv_1 + v + v_2, w) = cB(v_1, w) + B(v_2, w)$ and $B(v, cu_1 + u_2) = \overline{c}B(v, u_1) + B(v, u_2)$). Assume that for all $g \in G$ that $B(\rho_1(g)v, \rho_2(g)w) = B(v, w)$. If $B \neq 0$ then ρ_1 and ρ_2 are equivalent representations. (And thus is ρ_1 and ρ_2 are not equivalent then any such B is 0.)*

EXERCISE 4.1.16. Prove this. HINT: The map $w \mapsto B(\cdot, w)$ is a conjugate linear map from W to the space V^* of conjugate linear maps from V to $\mathbf{C}$. Now use Schur's lemma. □

If $\rho : G \to U(V)$ is a representation then a ***representative function*** for ρ any function on G of the form

$$f(g) := \langle \rho(g)v_1, v_2 \rangle$$

where v_1, v_2 are any elements of V.

PROPOSITION 4.1.17. *Let f_1 be a representative function for $\rho_1 : G \to U(V)$ and and f_2 be a representative function for $\rho_2 : G \to U(W)$. If ρ_1 and ρ_2 are irreducible and inequivalent then*

$$\int_G f_1(g)\overline{f_2(g)}\,dg = 0.$$

In particular if χ_{ρ_1} and χ_{ρ_2} are characters of inequivalent irreducible representations then they are orthogonal as elements of $L^2(G)$.

EXERCISE 4.1.18. Prove this. HINT: Choose $v_0 \in V$ and $w_0 \in W$ and define $B : V \times W \to \mathbf{C}$ by

$$B(v, w) = \int_G \langle \rho(g)v, v_0 \rangle \overline{\langle \rho(g)w, w_0 \rangle}\,dg.$$

and use the last proposition. □

COROLLARY 4.1.19. *If $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are irreducible representations of the compact Lie group G then the corresponding charters satisfy*

$$\int_G \chi_1(g)\overline{\chi_2(g)}\,dg = C(\rho_1)\delta_{\rho_1\rho_2} \tag{4.2}$$

where $C(\rho_1)$ is a positive constant only depending on ρ_1 and $\delta_{\rho_1\rho_2} = 1$ if ρ_1 and ρ_2 are equivalent and $\delta_{\rho_1\rho_2} = 0$ if ρ_1 and ρ_2 are inequivalent.

EXERCISE 4.1.20. Prove this. HINT: If ρ_1 and ρ_2 are inequivalent representations then the characters are sums of representative functions and the last proposition applies. If the two representations are equivalent then $\chi_1 = \chi_2$ and so $\chi_1(g)\overline{\chi_2(g)} = |\chi_1(g)|^2$ is a non-negative continuous function on G with $\chi_1(e) = \dim(V_1) > 0$ whose integral over G will thus be positive. □

PROPOSITION 4.1.21. *Let $\rho : G \to V = V_1 \oplus \cdots \oplus V_k$ be a direct sum of representations $\rho_i : G \to V_i$. Then the character of ρ is the sum of the characters of the ρ_i.*

EXERCISE 4.1.22. Prove this. □

PROPOSITION 4.1.23. *If χ_{ρ_1} and χ_{ρ_2} are characters of representations so are $\chi_{\rho_1} + \chi_{\rho_2}$, $\chi_{\rho_1}\chi_{\rho_2}$, and $\overline{\chi_{\rho_1}}$.*

EXERCISE 4.1.24. Prove this. HINT: Consider direct sums, tensor products, and the "conjugate dual" representation of all conjugate linear maps. □

EXERCISE 4.1.25. Prove Theorem 4.1.14. HINT: Given two representations ρ_1 and ρ_2, then write the corresponding characters χ_1 and χ_2 as the sum of irreducible characters. Then use the orthogonality relations (4.2) to show that each irreducible character appears in each sum the same number of times. □

4.2. The L^2 Convolution Algebra of a Compact Space

As we are assuming that the subgroup K is compact the homogeneous space G/K is compact if and only if the group G is compact. Recall that a compact group is unimodular and so the results of Theorem 3.3.2 apply. In this case we show that not only is $L^1(G;K)$ a Banach algebra, but so is $L^2(G;K)$. Toward this end let $L^2(G;K)$ be the set complex valued functions $h : G/K \times G/K \to \mathbf{C}$ so that $h(gx, gy) = h(x, y)$ with the norm

$$\|h\|_2^2 = \int_{G/K} |h(x, \mathbf{o})|^2 \, dx = \int_{G/K} |h(\mathbf{o}, x)|^2 \, dx \tag{4.3}$$

where the two integrals are equal by Theorem 3.3.2. This norm is a Hilbert space norm coming from the inner product

$$\langle p, q \rangle := \int_{G/K} p(x, \mathbf{o}) \overline{q(x, \mathbf{o})} \, dx = \int_{G/K} p(\mathbf{o}, y) \overline{q(\mathbf{o}, y)} \, dy \tag{4.4}$$

EXERCISE 4.2.1. Show that the two integrals defining $\langle p, q\rangle$ are equal and for any fixed point z_0 the inner product is also given by

$$\langle p, q \rangle = \int_{G/K} p(x, z_0) \overline{q(x, z_0)} \, dx = \int_{G/K} p(z_0, y) \overline{q(z_0, y)} \, dy.$$

HINT: The two integrals both define inner products and by (4.3) these two inner products have the same norm. Thus (4.4) follows by polarization. A change of variable in the integrals shows that $\mathbf{o}$ can be replaced by z_0. □

THEOREM 4.2.2. *If G/K is compact, then $L^2(G;K)$ is closed under the convolution product $(p, q) \mapsto p * q$ and for all $h \in L^2(G;K)$ the integral operator $T_h : L^2(G/K) \to L^2(G/K)$ is compact.*

PROOF. Let $p, q \in L^2(G;K)$ then by the Cauchy-Schwartz inequality

$$\begin{aligned}
\int_{G/K} |p * q(x, \mathbf{o})|^2 \, dx &\le \int_{G/K} \left(\int_{G/K} |p(x, z) q(z, \mathbf{o})| \, dz \right)^2 \, dx \\
&\le \int_{G/K} \int_{G/K} |p(x, z)|^2 \, dz \int_{G/K} |q(z, \mathbf{o})|^2 \, dz \, dx \\
&= \operatorname{Vol}(G/K) \|p\|_2^2 \|q\|_2^2 \qquad (4.5)
\end{aligned}$$

and G/K is compact and whence has finite volume. Thus $p * q$ is in $L^2(G;K)$ as claimed. If $h \in L^2(G;K)$ then

$$\int_{G/K \times G/K} |h(x, y)|^2 \, dx \, dy = \int_{G/K} \|h\|_2^2 \, dy = \operatorname{Vol}(G/K) \|h\|_2^2$$

which implies the integral operator T_h is a Hilbert-Schmidt operator and thus compact by Proposition A.2.4. $\square$

CHAPTER 5

Compact Symmetric and Weakly Symmetric Spaces

In this section we will assume that the functions in $L^p(G;K)$ are all real valued. Recall (Theorem 3.4.3) that the convolution algebra $L^1(G;K)$ of a weakly symmetric space is commutative. So in light of the results above:

PROPOSITION 5.0.3. *If G/K is a compact weakly symmetric space then the space $L^2(G;K)$ with the product $*$ is a commutative Banach algebra. If $T_h f(x) = \int_{G/K} h(x,y)f(y)\,dy$ then the set $\mathcal{A} := \{T_h : h \in L^p(G;K)\}$ is a algebra of commuting compact self-adjoint linear operators on $L^2(G;K)$.*

PROOF. That $L^2(G;K)$ is commutative follows form theorem 3.4.3. The rest follows form theorem 4.2.2. $\square$

Let Ψ be the set of all non-zero ***weights*** of $L^2(G;K)$ on $L^2(G/K)$. That is $\mathcal{A}$ is the set of all non-zero linear functions $L^2(G;K) \to \mathbf{R}$ so that the ***weight space***

$$E_\alpha := \{f : T_h f = \alpha(h) f \quad \text{for all } h \in L^2(G;K)\} \tag{5.1}$$

is nonzero. If E is any G-submodule of $L^2(G/K)$ then set

$$E^K := \{f \in E : \tau_a f = f \quad \text{for all} \quad a \in K\} = \text{set of isotropic functions in } E.$$

5.1. The Decomposition of $L^2(G/K)$ for Weakly Symmetric Spaces

THEOREM 5.1.1. *Let G/K be a compact weakly symmetric space and Ψ be the set of non-zero weights of $L^2(G;K)$ on $L^2(G/K)$. Then*

1. *Each E_α is a G-submodule of $L^2(G/K)$ and*
$$L^2(G/K) = \bigoplus_{\alpha\in\Psi} E_\alpha \qquad \textit{(Orthogonal direct sum).}$$
2. *Each E_α is finite dimensional and consists of C^∞ functions.*
3. *Each E_α is an irreducible G-module.*
4. *If $\alpha \neq \beta$ then E_α and E_β are not isomorphic as G-modules.*
5. *Each E^K_α is one dimensional and spanned by a unique element p_α with $p_\alpha(\mathbf{o}) = 1$. This function is called the **spherical function** in E_α.*

6. *If $E \subseteq L^2(G/K)$ is a closed G-submodule then for some subset $A \subseteq \Psi$*

$$E = \bigoplus_{\alpha \in A} E_\alpha.$$

If E is finite dimensional then the number of irreducible factors in the direct sum is $\dim E^K$. Thus E is irreducible if and only if $\dim E^K = 1$. In particular if E is an irreducible submodule of $L^2(G/K)$, then $E = E_\alpha$ for some $\alpha \in \Psi$.

STEP 1. Parts 1 and 2 of the theorem hold.

PROOF. If $f \in E_\alpha$ then, using that the linear operators T_h with $h \in L^2(G;K)$ commute with the action of G,

$$T_h \tau_g f = \tau_f T_h f = \tau_g \alpha(h) f = \alpha(h) \tau_g f.$$

Thus $\tau_g f \in E_\alpha$ so E_α is a G-submodule. Let $E_0 = \{f : T_h f = 0 \quad \text{for all} \quad h \in L^2(G;K)\}$. By the spectral theorem for commuting compact self-adjoint linear maps on a Hilbert space (Theorem A.2.1) applied to the family $\{T_h : h \in L^2(G;K)\}$

$$L^2(G/K) = E_0 \oplus \bigoplus_{\alpha \in \Psi} E_\alpha.$$

So to finish the proof of Step 1 it is enough to show that $E_0 = \{0\}$. Let $\Phi_\delta \in L^2(G;K)$ be as in Theorem 3.3.5. Then $T_{\Phi_\delta} f = 0$ as $f \in E_0$, but theorem 3.3.5 implies

$$f = \lim_{\delta \downarrow 0} T_{\Phi_\delta} f = 0.$$

To see $f \in E_\alpha$ must be in $C^\infty(G/K)$ note for $f \in E_\alpha$ that $f = \lim_{\delta \downarrow 0} T_{\Phi_\delta} f = \alpha(\Phi_\delta) f$ and thus $\lim_{\delta \downarrow 0} \alpha(\Phi_\delta) = 1$. So for small δ, $\alpha(\delta) \neq 0$. But then

$$\alpha(\Phi_\alpha) f(x) = T_{\Phi_\delta} f(x) = \int_{G/K} \Phi_\delta(x,y) f(y)\, dy$$

By differentiating under the integral we see that $f \in C^\infty$. $\square$

STEP 2. If $\{0\} \neq E \subset C(G/K)$ (the continuous functions on G/K) is any finite dimensional G submodule, then there is a $p \in E^K$ with $p(\mathbf{o}) = 1$.

PROOF. As the functions in E are continuous there is a well defined evaluation map $e : E \to \mathbf{R}$ given by $e(f) := f(\mathbf{o})$. As G/K is compact $C(G/K) \subset L^2(G/K)$ so again using that E is finite dimensional every linear function on E can be uniquely represented as an inner product. That is there is a unique function $p_0 \in E$ so that for all $f \in E$

$$f(\mathbf{o}) = \int_{G/K} p_0(x) f(x)\, dx. \tag{5.2}$$

If $a \in K$ and $f \in E$ then using that the measure dx is invariant and the last equality

$$\begin{aligned}\int_{G/K} (\tau_a p)(x) f(x)\, dx &= \int_{G/K} p(a^{-1}x) f(x)\, dx \\ &= \int_{G/K} p(x) f(ax)\, dx = f(a\mathbf{o}) = f(\mathbf{o})\end{aligned}$$

so by the uniqueness of the element representing the functional e we see $\tau_a p = p$ for all $a \in K$. Thus p is isotropic. As the action of G is transitive and $E \neq \{0\}$ there are functions $f \in E$ with $f(\mathbf{o}) \neq 0$ which by (5.2) implies $p_0 \neq 0$. Using $f = p_0$ in (5.2)

$$p_0(\mathbf{o}) = \int_{G/K} p_0(x)^2\, dx > 0.$$

Letting $p = \frac{1}{p_0(\mathbf{o})} p_0$ completes the proof. $\square$

STEP 3. Each E_α^K is one dimensional. Thus Part 5 of the theorem holds.

PROOF. Form the last step we know there is a $p_\alpha \in E_\alpha^K$ with $p_\alpha(\mathbf{o}) = 1$. Let $h_0 \in E_\alpha^K$ and set

$$h = h_0 - h_0(\mathbf{o}) p_\alpha.$$

Then

$$h(\mathbf{o}) = h_0(\mathbf{o}) - h_0(\mathbf{o}) p_\alpha(\mathbf{o}) = 0.$$

If we can show $h = 0$ then $h_0 = h_0(\mathbf{o}) p_\alpha$ and thus p_α spans E_α^K.

As h is isotropic we can use proposition 3.4.5 to define an element $H \in C(G;K) \subset L^2(G;K)$ by $H(x,y) = (\operatorname{Ext} h)(x,y) = h(\xi^{-1}y)$ where $\xi \in G$ is any element with $\xi\mathbf{o} = x$. Form the definition of E_α for any $f \in E_\alpha$

$$\alpha(H) f(x) = T_H f(x) = \int_{G/K} h(\xi^{-1}y) f(y)\, dy, \quad \xi \in G,\ \xi\mathbf{o} = x.$$

Let $f = h$, $x = \mathbf{o}$ (in which case we can use $\xi = e$) and using $h(\mathbf{o}) = 0$

$$0 = \alpha(H) h(\mathbf{o}) = \int_{G/K} h(y)^2\, dy.$$

As h is continuous this implies $h = 0$ and completes the proof $\square$

STEP 4. Each E_α^K is irreducible. Thus Part 3 of the theorem holds.

PROOF. If E_α is not irreducible then if can be decomposed as a direct sum $E_\alpha = E_1 \oplus E_2$ with each E_i a nontrivial G submodule. By step 2 each of E_1^K and E_2^K is at least one dimensional and therefore $E_\alpha^K \supseteq E_1^K \oplus E_2^K$ is at least two dimensional which contradicts step 3. $\square$

STEP 5. Let $f_1, f_2 \in L^2(G/K)$. Then for each $\alpha \in \Psi$ there is a constant $c_\alpha(f_1, f_2)$ so that

$$\int_{G/K} \int_G f_1(g^{-1}x) f_2(g^{-1}y)\, dg f(y)\, dy = c_\alpha(f_1, f_2) f(x) \tag{5.3}$$

PROOF. Write the left hand side of this equation as

$$\int_{G/K} h(x, y) f(y)\, dy$$

where

$$h(x, y) = \int_G f_1(g^{-1}x) f_2(g^{-1}y)\, dg.$$

We may assume that G has a Riemannian metric that is adapted to the metric of G/K in the sense of proposition 2.3.14. Note by the Cauchy-Schwartz inequality and the formulas of proposition 2.3.15

$$\begin{aligned}
\int_{G/K} |h(x, \mathbf{o})|^2\, dx &= \int_{G/K} \left| \int_G f_1(g^{-1}x) f_2(g^{-1}\mathbf{o})\, dg \right|^2 dx \\
&\le \int_{G/K} \int_G f_1(g^{-1}x)^2\, dg \int_G f_2(g^{-1}y)^2\, dg\, dx \\
&= \operatorname{Vol}(G/K) \operatorname{Vol}(K)^2 \|f_1\|_{L^2}^2 \|f_2\|_{L^2}^2 \\
&< \infty
\end{aligned}$$

and likewise $\int_{G/K} |h(\mathbf{o}, y)|^2\, dy \le \operatorname{Vol}(G/K) \operatorname{Vol}(K)^2 \|f_1\|_{L^2}^2 \|f_2\|_{L^2}^2 < \infty$. If $\xi \in G$ then

$$\begin{aligned}
h(\xi x, \xi y) &= \int_G f_1(g^{-1}\xi x) f_2(g^{-1}\xi y)\, dg \\
&= \int_G f_1(g^{-1}x) f_2(g^{-1}y)\, dg \qquad \text{(change of variable } g \mapsto \xi g) \\
&= h(x, y).
\end{aligned}$$

Thus $h \in L^2(G; K)$. As $f \in E_\alpha$

$$\int_{G/K} h(x, y) f(y)\, dy = \alpha(h) f(x).$$

Whence the result holds with $c_\alpha(f_1, f_2) = \alpha(h)$. □

STEP 6. If $\alpha, \beta \in \Psi$ and $\alpha \neq \beta$ then E_α is not equivalent to E_β as a G-module. Thus Part 4 of the theorem holds.

PROOF. Let

$$\tau_\alpha = \tau_g|_{E_\alpha} \qquad \tau_\beta = \tau_g|_{E_\beta}$$

be the induced representations on E_α and E_β. Let $\chi_\alpha(g) = \operatorname{trace}(\tau_\alpha(g))$ and $\chi_\beta(g) = \operatorname{trace}(\tau_\beta(g))$ be the corresponding characters. By Proposition 3.1.4 $\chi_\alpha = \chi_\beta$.

Choose orthonormal basis $f_{\alpha 1}, \dots, f_{\alpha l}$ and $f_{\beta 1}, \dots, f_{\beta m}$ of E_α and E_β. In the basis $f_{\alpha 1}, \dots, f_{\alpha m}$ the matrix representing $\tau_\alpha(g)$ is $[\langle \tau_g f_{\alpha i}, f_{\alpha j}\rangle]$ and the trace is the sum of the diagonal elements of the matrix. Thus

$$\begin{aligned}\chi_\alpha(g) = \operatorname{trace}(\tau_\alpha(g)) &= \sum_{i=1}^{l} \langle \tau_\alpha(g) f_{\alpha i}, f_{\alpha i}\rangle \\ &= \sum_{i=1}^{l} \int_{G/K} f_{\alpha i}(g^{-1}x) f_{\alpha i}(x)\, dx\end{aligned}$$

and likewise

$$\chi_\beta(g) = \sum_{j=1}^{m} \int_{G/K} f_{\beta j}(g^{-1}y) f_{\beta j}(y)\, dy.$$

Using these relations and interchanging the order of integration

$$\begin{aligned}&\int_G \chi_\alpha(g)\chi_\beta(g)\, dg \\ &\quad = \sum_{i,j} \int_{G/K} \left(\int_{G/K} \int_G f_{\alpha i}(g^{-1}x) f_{\beta j}(g^{-1}y)\, dg f_{\beta j}(y)\, dy \right) f_{\alpha i}(x)\, dx \\ &\quad = \sum_{i,j} c_\beta(f_{\alpha i}, f_{\beta j}) \int_{G/K} f_{\beta j}(x) f_{\alpha i}(x)\, dx \qquad \text{(by step 5)} \\ &\quad = 0\end{aligned}$$

where $\int_{G/K} f_{\beta j}(x) f_{\alpha i}(x)\, dx = 0$ as E_α and E_β are orthogonal. But if E_α and E_β are isomorphic then $\chi_\alpha = \chi_\beta$ this leads to the contradiction $0 = \int_G \chi_\alpha(g)\chi_\beta(g)\, dg = \int_G \chi_\alpha(g)^2\, dg > 0$. (Note $\chi_\alpha(e) = \dim E_\alpha$ so $\chi_\alpha \neq 0$.) This completes the proof. □

STEP 7. If $\{0\} \neq E \subset L^2(G/K)$ is a finite dimensional irreducible G-module, then $E = E_{\alpha_0}$ for some α_0.

PROOF. Let $P_\alpha : L^2(G/K) \to E_\alpha$ be the orthogonal projection of $L^2(G/K)$ onto E_α. Then as both E_α and its orthogonal complement $E_\alpha^\perp$ are invariant under the action of G the map P_α is a G-map. If $P_\alpha E = \{0\}$ for all α then $E = \{0\}$ as $L^2(G/K) = \bigoplus_{\alpha\in\Psi} E_\alpha$. Thus for some α_0, $P_{\alpha_0} E_{\alpha_0} \neq \{0\}$. The map $P_{\alpha_0}|_E : E \to E_{\alpha_0}$ is a nonzero intertwining map, thus by Schur's lemma (proposition 3.1.1) $P_{\alpha_0}|_E : E \to E_{\alpha_0}$ is an isomorphism. Thus E is isomorphic to E_{α_0} as a G-module. If $\alpha \neq \alpha_0$ then by the last step E_α and E_{α_0} are not isomorphic as G-modules and thus Schur's lemma implies that $P_\alpha|_E : E \to E_\alpha$ is the zero map for $\alpha \neq \alpha_0$. This implies $E \subseteq E_{\alpha_0}$. An as E is a nonzero submodule and E_{α_0} irreducible $E = E_{\alpha_0}$. □

STEP 8. If E is a closed G-submodule of $L^2(G/K)$ then for some $A \subseteq \Psi$ $E = \bigoplus_{\alpha\in\Psi} E_\alpha$.

PROOF. Let $P : L^2(G/K) \to E$ be the orthogonal projection. Then P is a G-map. By Schur's lemma for each α with $PE_\alpha \neq \{0\}$ PE_α is a G submodule isomorphic to E_α and so by the last step $E_\alpha = PE_\alpha \subset \operatorname{Image} P = E$. If $PE_\alpha = \{0\}$ then $E_\alpha \subseteq E^\perp$. Therefore for each α either $E_\alpha \subset E$ or $E_\alpha \subseteq E^\perp$. As $L^2(G/K) = \bigoplus_{\alpha \in \Psi} E_\alpha$ it follows that $E = \bigoplus_{\alpha \in A} E_\alpha$ where $A = \{\alpha \in \Psi : PE_\alpha \neq \{0\}\}$. □

STEP 9. If E is a finite dimensional G-submodule of $L^2(G/K)$ then the number of irreducible components in E is $\dim E^K$. This completes the proof to the theorem.

PROOF. If E is finite dimensional it is closed in $L^2(G/K)$ and thus thus by the last step there is a finite set $\{\alpha_1, \dots, \alpha_l\} \subseteq \Psi$ so that

$$E = E_{\alpha_1} \oplus \cdots \oplus E_{\alpha_l}.$$

It follows easily that

$$E^K = E^K_{\alpha_1} \oplus \cdots \oplus E^K_{\alpha_l}.$$

But by Part 5 of the theorem each $E^K_{\alpha_i}$ is one dimensional which finishes the proof. □

5.2. Diagonalization of Invariant Linear Operators on Compact Weakly Symmetric spaces

This this section G/K will always be a compact weakly symmetric space and we will use the notation of theorem 5.1.1.

THEOREM 5.2.1. *Let G/K be a compact weakly symmetric space and let $\mathcal{D} \subseteq L^2(G/K)$ be a G-invariant subspace on that contains all of the subspaces E_α. (For example $\mathcal{D} = C^\infty(G/K)$ or $\mathcal{D} = C(G/K)$. It is not assume that $\mathcal{D}$ is closed.) Let $L : \mathcal{D} \to L^2(G/K)$ be an invariant operator (in the sense that $\tau_g \circ L = L \circ \tau_g$ and which need not be continuous). Then for each $\alpha \in \Psi$*

$$LE_\alpha \subseteq E_\alpha \tag{5.4}$$

and if p_α is the spherical function in E_α then for $f \in E_\alpha$

$$Lf = (Lp_\alpha)(\mathbf{o})f \tag{5.5}$$

so formally the operator L is

$$L = \bigoplus_{\alpha \in \Psi} (Lp_\alpha)(\mathbf{o}) \operatorname{Id}_{E_\alpha}. \tag{5.6}$$

PROOF. If $LE_\alpha = \{0\}$ then $LE_\alpha \subseteq E_\alpha$ and (5.5) and (5.6) hold. Thus assume that $LE_\alpha \neq \{0\}$. Then by Schur's lemma $L|_{E_\alpha} \to LE_\alpha$ is an isomorphism. Thus by Parts 4 and 6 of theorem 5.1.1 this implies $LE_\alpha = E_\alpha$. As the operator L is invariant it maps isotropic functions to isotropic functions and thus $Lp_\alpha \in E^K_\alpha$. By Part 5 of theorem 5.1.1 E^K_α is one dimensional and thus $Lp_\alpha = cp_\alpha$ for some $c \in \mathbf{R}$. Therefore $\ker(L|_{E_\alpha} - c \operatorname{Id}_{E_\alpha}) \neq \{0\}$. But $\ker(L|_{E_\alpha} - c \operatorname{Id}_{E_\alpha})$ is a G-submodule of E_α and by Part 3 of theorem 5.1.1

$_\alpha$ is irreducible. Therefore $\ker(L|_{E_\alpha} - c\,\mathrm{Id}_{E_\alpha}) = E_\alpha$ and thus $L|_{E_\alpha} = c\,\mathrm{Id}_{E_\alpha}$. To compute c use that $p_\alpha(\mathbf{o}) = 1$: $c = cp_\alpha(\mathbf{o}) = (Lp_\alpha)(\mathbf{o})$. This shows that (5.5) holds and completes the proof. □

5.3. Abelian Groups and Spaces with Commutative Convolution Algebra

The proof structure theorem of Section 5.1 really only used that the convolution algebra $L^2(G;K)$ was commutative. Here we state the more general result leaving most of the proof as exercises with hints. In this generality the theory also applies directly to compact Abelian groups and in particular to $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ where the expansion of a function $f \in L^2(S^1)$ as $f(\theta) = \sum_{k=\infty}^{\infty} c_k e^{k\sqrt{-1}\theta}$ is the basic example for much of classical harmonic analysis.

Let G/K be a compact homogeneous space and let $C^\infty(G;K)$ be the space of smooth (i.e C^∞) complex valued functions h so that $h(gx, gy) = h(x,y)$ for all $g \in G$. As is the real valued case this is closed under the convolution product

$$h * k(x,y) := \int_{G/K} h(x,z)k(z,y)\,dz.$$

Let $L^2(G/K)$ be the Hilbert space of complex valued function $f : (G/K) \to \mathbf{C}$ with the Hermitian inner product

$$\langle f_1, f_2\rangle := \int_{G/K} f_1(x)\overline{f_2(x)}\,dx.$$

As before for each $h \in C^\infty(G;K)$ define a linear operator

$$T_h f(x) := \int_{G/K} h(x,y)\,dy.$$

If $h \in C^\infty(G;K)$ then set $h^*(x,y) = h(y,x)$. Then T_{h^*} is the adjoint of T_h in the sense that

$$\langle T_h f_1, f_2\rangle = \langle f_1, T_{h^*} f_2\rangle.$$

Thus in the complex valued case symmetry $h(x,y) = h(y,x)$ does not imply T_h is selfadjoint.

A ***weight*** for $C^\infty(G;K)$ is a linear functional $\alpha : C^\infty(G;K) \to \mathbf{C}$ so that corresponding ***weight space***

$$E_\alpha := \{f \in L^2(G/K) : T_h f = \alpha(h)\alpha(f)\}.$$

is not the zero space $\{0\}$. If $E \subseteq L^2(G/K)$ is a G-submodule then the set of isotropic functions E^K in E is

$$E^K := \{f \in E : f(ax) = f(x) \text{ for all } a \in K\}.$$

We know that if the space G/K is weakly symmetric then the convolution algebra $C^\infty(G;K)$ is commutative. There are other cases where this holds. For example let G be compact and commutative and let $K = \{e\}$. Then $h \in C^\infty(G;\{e\})$ if and only if it is of the form $h(x,y) = f(xy^{-1})$ for some smooth complex valued $f : G \to \mathbf{C}$. From this it is not hard to show $C^\infty(G; \{e\})$ is commutative. As a first step toward understanding Fourier expansions on compact groups prove the following variant of our basic result about the decompositions of $L^2(G/K)$ when G/K is weakly symmetric.

THEOREM 5.3.1. *Let G/K be a compact homogeneous space so that the convolution algebra $C^\infty(G; K)$ is commutative. Let Ψ be the set of non-zero weights of $C^\infty(G; K)$ on $L^2(G/K)$. Then*

1. *Each E_α is a G-submodule of $L^2(G/K)$ and*
$$L^2(G/K) = \bigoplus_{\alpha\in\Psi} E_\alpha \qquad \textit{(Orthogonal direct sum).}$$
2. *Each E_α is finite dimensional and consists of C^∞ functions.*
3. *Each E_α is an irreducible G-module.*
4. *If $\alpha \neq \beta$ then E_α and E_β are not isomorphic as G-modules.*
5. *Each E_α^K is one dimensional and spanned by a unique element p_α with $p_\alpha(\mathbf{o}) = 1$. This function is called the* ***spherical function*** *in E_α.*
6. *If $E \subseteq L^2(G/K)$ is a closed G-submodule then for some subset $A \subseteq \Psi$*
$$E = \bigoplus_{\alpha\in A} E_\alpha.$$
If E is finite dimensional then the number of irreducible factors in the direct sum is $\dim E^K$. Thus E is irreducible if and only if $\dim E^K = 1$. In particular if E is an irreducible submodule of $L^2(G/K)$, then $E = E_\alpha$ for some $\alpha \in \Psi$.

EXERCISE 5.3.2. Prove this theorem. HINT: The basic analytic tool in the case of weakly symmetric spaces was the spectral theorem for commuting compact self-adjoint operators on a Hilbert space. In the case at hand the operators T_h are no longer selfadjoint but this is not a large problem as if $C^\infty(G; K)$ is commutative, then the operator T_h commutes with its adjoint T_{h^*}. That is T_h is a normal operator and commuting compact normal operators have a spectral theory every bit as nice as commuting compact selfadjoint operators. See Theorem A.2.2 in the appendix. Now go through the proof of Theorem 5.1.1 and make the (mostly straightforward) changes needed to prove result here. $\square$

Now the proof of the diagonalization result Theorem 5.2.1 goes through just as before:

THEOREM 5.3.3. *Let G/K be a compact homogeneous space so that the convolution algebra $C^\infty(G; K)$ is commutative and let $\mathcal{D} \subseteq L^2(G/K)$ be a G-invariant subspace on that contains all of the subspaces E_α. (For example $\mathcal{D} = C^\infty(G/K)$ or $\mathcal{D} = C(G/K)$. It is not assume that $\mathcal{D}$ is closed.) Let $L : \mathcal{D} \to L^2(G/K)$ be an invariant operator (in the sense that $\tau_g \circ L = L \circ \tau_g$*

and which need not be continuous). Then for each $\alpha \in \Psi$

$$LE_\alpha \subseteq E_\alpha \tag{5.7}$$

and if p_α *is the spherical function in* E_α *then for* $f \in E_\alpha$

$$Lf = (Lp_\alpha)(\mathbf{o})f \tag{5.8}$$

so formally the operator L *is*

$$L = \bigoplus_{\alpha\in\Psi} (Lp_\alpha)(\mathbf{o})\,\mathrm{Id}_{E_\alpha}\,. \tag{5.9}$$

EXERCISE 5.3.4. Prove this by making the required modifications to the proof of Theorem 5.2.1. □

5.3.1. Compact Abelian Groups. In this section we specialize the results above to the case of G compact and Abelian. Form the point of view of representation theory the next result shows how compact Abelian groups differ form general compact groups.

PROPOSITION 5.3.5. *Any finite dimensional complex irreducible representation of a compact Abelian group* G *is one dimensional. Any real irreducible representation of* G *is either one or two dimensional.*

EXERCISE 5.3.6. Prove this. HINT: One method (and this is really using overkill) is to note that if $\rho : G \to GL(V)$ is a finite dimensional representation of G then (after using Proposition 4.1.1 to get an invariant inner product on V) the image $\rho[G]$ will be a commuting set of unitary (and thus also normal) maps. Therefore the spectral theorem for commuting compact normal operators A.2.2 can be used to show that V decomposes into one dimensional invariant subspaces. □

When G is Abelian any subgroup K is normal and so G/K is also a compact Abelian group and we do not lose anything by replacing G by G/K and assuming $K = \{e\}$ is the unit subgroup.

THEOREM 5.3.7. *Let* G *be a compact Abelian Lie group and let* Ψ *the the nonzero weights of* $C^\infty(G;\{e\})$ *on* $L^2(G/K)$ *and*

$$L^2(G) = \bigoplus_{\alpha\in\Psi} E_\alpha$$

the corresponding decomposition of $L^2(G/K) = L^2(G)$*. Then*

1. *Each* E_α *is one dimensional and thus* $E_\alpha = E_\alpha^K$*.*
2. *If* χ_α *is the spherical function in* E_α *(which will be the unique element of* E_α *with* $\chi_\alpha(\mathbf{o}) = 1$*), then* χ_α *is a group homomorphism* $\chi_\alpha : G \to T^1$*. (Here* $T^1 := \{z \in \mathbf{C} : |z| = 1\}$ *is the group of complex numbers of modulus one.)*
3. *If* $\chi : G \to T^1$ *is a continuous group homomorphism then* $\chi = \chi_\alpha$ *for some* $\alpha \in \Psi$*.*

4. *If* $\alpha \neq \beta$, *then*
$$\langle \chi_\alpha, \chi_\beta \rangle = \int_G \chi_\alpha(x)\overline{\chi_\beta(x)}\, dx = 0$$
5. *Every* $f \in L^2(G)$ *has an expansion*
$$f = \sum_{\alpha \in \Psi} c_\alpha(f)\chi_\alpha \qquad \textit{where} \quad c_k(f) := \frac{1}{\mathrm{Vol}(G)} \int_G f(x)\overline{\chi_\alpha(x)}\, dx$$
6. *Make* $L^2(G)$ *into a Banach algebra with the product* $f_1 \star f_2(x) := \int_G f_1(xy^{-1}) f_2(y)\, dy$. *Then the maps that sends* $f \in L^2(G)$ *to* $h(x, y) := f(xy^{-1})$ *is an algebra isomorphism of* $(L^2(G), \star)$ *and* $(L^2(G; K), *)$.

EXERCISE 5.3.8. Prove this by showing it is a special case of Theorem 5.3.1. $\square$

APPENDIX A

Some Results from Analysis

A.1. Bounded Integral Operators

First we give a useful result about when certain integral operators on L^p spaces are bounded. Let (X, μ), and (Y, ν) be sigma finite measure spaces. Let $K : X \times Y \to \mathbf{R}$ be a measurable function. Let $L^+(X, \mu)$ be the set of non-negative measurable functions define on X (where the value ∞ is permitted). Then define $P_K : L^+(X, \mu) \to L^+(Y, \nu)$ and $P_K^* : L^+(Y, \nu) \to L^+(X, \mu)$ by

$$
\begin{aligned}
P_K f(y) &= \int_X |K(x,y)| f(x)\, d\mu(x)\\
P_K^* f(x) &= \int_Y |K(x,y)| f(y)\, d\nu(y)
\end{aligned}
$$

Theorem A.1.1. *Let $1 \le p < \infty$ Assume there is a positive function $h \in L^+(X, \mu)$ and a number $\lambda > 0$ so that*

$$P_K^*(P_K h)^{p-1} \le \lambda h^{p-1} \tag{A.1}$$

Then the integral operator T_K defined by

$$T_K f(y) := \int_X K(x,y) f(x)\, d\mu(x) \tag{A.2}$$

is a bounded linear map $T_K : L^p(X, \mu) \to L^p(Y, \nu)$ and

$$\|T_K f\|_{L^p} \le \lambda^{\frac{1}{p}} \|f\|_{L^p} \tag{A.3}$$

Remark A.1.2. The function h need not be in L^p. Being able to choose the function h with it having to be in some L^p space is what makes the result useful.

PROOF. Let $p' = p/(p-1)$ so that $1/p + 1/p' = 1$ and $p/p' = p-1$. Thus by Hölder's inequality for any $f \in L^p(X,\mu)$

$$
\begin{aligned}
|T_K f(y)| &\le \int_X |K(x,y)||f(x)|\,d\mu(x) \\
&= \int_X |K(x,y)|^{\frac{1}{p'}} h(x)^{\frac{1}{p'}} |K(x,y)|^{\frac{1}{p}} h(x)^{\frac{-1}{p'}} |f(x)|\,d\mu(x) \\
&\le \left(\int_X |K(x,y)| h(x)\,d\mu(x)\right)^{\frac{1}{p'}} \left(\int_X |K(x,y)| h(x)^{-(p-1)} |f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}} \\
&= ((P_K h)(y))^{\frac{1}{p'}} \left(\int_X |K(x,y)| h(y)^{-(p-1)} |f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}}
\end{aligned}
$$

That is

$$
|T_K f(y)|^p \le ((P_K h)(y))^{p-1} \int_X |K(x,y)| h(x)^{-(p-1)} |f(x)|^p\,d\mu(x)
$$

Therefore

$$
\begin{aligned}
\|T_K f\|_{L^p}^p &= \int_Y |T_K(y)|^p\,d\nu(y) \\
&\le \int_Y ((P_K h)(y))^{p-1} \int_X |K(x,y|h(x)|^{-(p-1)} |f(x)|^p\,d\mu(x)\,d\nu(x) \\
&= \int_X \int_Y |K(x,y)|\,((P_K h)(y))^{p-1}\ d\nu(y)\,h(x)^{-(p-1)} |f(x)|^p\,d\mu(x) \\
&= \int_X \left(P_K^*(P_K h)^{p-1}\right)(x) h(x)^{-(p-1)} |f(x)|^p\,d\mu(x) \\
&\le \lambda \int_X h(x)^{p-1} h(x)^{-(p-1)} |f(x)|^p\,d\mu(x) \\
&= \lambda \|f\|_{L^p}^p.
\end{aligned}
$$

□

COROLLARY A.1.3. *If $K : X \times Y \to \mathbf{R}$ satisfies*

$$
\int_X |K(x,y)|\,d\mu(x) \le A, \qquad \int_Y |K(x,y)|\,d\nu(y) \le B \tag{A.4}
$$

for constants A, B. Then for $1 \le p < \infty$ the integral operator T_K is bounded and a map from $L^p(X,\mu)$ to $L^p(Y,\nu)$ and

$$
\|T_K f\|_{L^p} \le A^{\frac{1}{p}} B^{\frac{p-1}{p}} \|f\|_{L^p} \tag{A.5}
$$

PROOF. Let $h \equiv 1$ in the last theorem. Then the bounds (A.4) imply $P_K 1 \le A$ and so $P_K^*(P_K 1)^{p-1} \le A^{p-1} B\,1$. Thus let $\lambda = A^{p-1}B$ and use the theorem. □

Exercise A.1.4. As an example of the use of Theorem A.1.1 define the Hardy operator on functions defined on $(0,\infty)$ by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)\,dt.$$

Show that for $1 < p < \infty$ that $H : L^p(0,\infty) \to L^p(0,\infty)$ is bounded linear map and that

$$\|Hf\|_{L^p} \le \frac{p}{p-1}\|f\|_{L^p}.$$

Hint: In this case $P_K = H$ and $P_K^* f(x) = \int_x^\infty \frac{f(t)}{t}\,dt$. Let $h_\alpha(x) = t^\alpha$ where $-1 < \alpha < 0$ and show $P_K^*(P_K h_\alpha)^{p-1} = \lambda(\alpha)h_\alpha$ with $\lambda(\alpha) = -1/(\alpha(\alpha+1)^{p-1}(p-1))$. Now make a smart choice of α. □

A.2. Spectral Theorem for Commuting Compact Selfadjoint and Normal Operators on a Hilbert Space

Let $\mathcal{H}$ be a real or complex Hilbert space with inner product $\langle\cdot\,,\cdot\rangle$. Recall that a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ is ***self-adjoint*** or ***symmetric*** iff $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}$. The linear map A is ***compact*** iff $A[B(0,1)]$ has compact closure in $\mathcal{H}$. ($B(0,1)$ is the unit ball about the origin in $\mathcal{H}$.) This A is compact iff for any bounded sequence $\{x_n\}_{n=1}^\infty$ from $\mathcal{H}$ the sequence $\{Ax_n\}_{n=1}^\infty$ has a convergent subsequence.

Let $\mathcal{A}$ be a linear space of compact self-adjoint linear operators on $\mathcal{H}$. (Note that even when the space $\mathcal{H}$ is complex the space $\mathcal{A}$ will be a real vector space as the set of self-adjoint operators is not closed under multiplication by $\sqrt{-1}$.) A linear map $\alpha : \mathcal{A} \to \mathbf{R}$ is a ***weight*** iff the corresponding ***weight space***

$$E_\alpha := \bigcap_{A\in\mathcal{A}} \ker(A - \alpha(A)) = \{x \in \mathcal{H} : Ax = \alpha(A)x \quad \text{for all} \quad A \in \mathcal{A}\} \tag{A.6}$$

is not the trivial subspace $\langle 0\rangle$.

Theorem A.2.1 (Spectral theorem for compact selfadjoint operators). *Let $\mathcal{A}$ be a vector space of compact selfadjoint linear maps on the Hilbert space $\mathcal{H}$ and assume that any two elements of $\mathcal{A}$ commute. Let Ψ be the set of non-zero weights of $\mathcal{A}$. Then there is an orthogonal direct sum decomposition of $\mathcal{H}$ given by*

$$\mathcal{H} = E_0 \oplus \bigoplus_{\alpha\in\Psi} E_\alpha$$

where for each $\alpha \in \Psi$ each space E_α is finite dimensional. (However the subspace $E_0 := \{x : Ax = 0 \text{ for all } A \in \mathcal{A}\}$ can be infinite dimensional.) □

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot\,,\cdot\rangle$. Recall that a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ is ***normal*** iff A commutes with its adjoint A^* (i.e. $A^*A = AA^*$ and where the adjoint A^* of A is defined by $\langle Ax, y\rangle = \langle x, A^*y\rangle$). The linear map A is ***compact*** iff $A[B(0,1)]$ has compact closure in $\mathcal{H}$. ($B(0,1)$ is the unit ball about the origin in $\mathcal{H}$.) This

A is compact iff for any bounded sequence $\{x_n\}_{n=1}^{\infty}$ from $\mathcal{H}$ the sequence $\{Ax_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Let $\mathcal{A}$ be a linear space of compact normal linear operators on $\mathcal{H}$.

$\alpha : \mathcal{A} \to \mathbf{C}$ is a ***weight*** iff the corresponding ***weight space***

$$E_\alpha := \bigcap_{A \in \mathcal{A}} \ker(A - \alpha(A)) = \{x \in \mathcal{H} : Ax = \alpha(A)x \quad \text{for all} \quad A \in \mathcal{A}\}$$

is not the trivial subspace $\{0\}$.

THEOREM A.2.2 (Spectral theorem for commuting compact normal operators). *Let $\mathcal{A}$ be a vector space of compact normal linear maps on the Hilbert space $\mathcal{H}$ and assume that any two elements of $\mathcal{A}$ commute. Let Ψ be the set of non-zero weights of $\mathcal{A}$. Then there is an orthogonal direct sum decomposition of $\mathcal{H}$ given by*

$$\mathcal{H} = E_0 \oplus \bigoplus_{\alpha \in \Psi} E_\alpha$$

where for each $\alpha \in \Psi$ each space E_α is finite dimensional. (However the subspace $E_0 := \{x : Ax = 0 \text{ for all } A \in \mathcal{A}\}$ can be infinite dimensional.) □

REMARK A.2.3. This can be reduced to the case of the Spectral Theorem A.2.1 for compact self-adjoint operators. Here is an out line of how to reduce this to the selfadjoint case. If A is any operator on a Hilbert space, then write $A = U + \sqrt{-1}V$ where $U = U(A) := \frac{1}{2}(A + A*)$ and $V = V(a) := \frac{1}{2\sqrt{-1}}(A - A^*)$. Then U and V are self-adjoint and if A is normal then U and V commute. Also if A is compact, then so are U and V. Thus $\mathcal{B} := \operatorname{Span}\{U(A), V(A) : A \in \mathcal{A}\}$ is a linear space of commuting self-adjoint compact operators. This use the spectral in the self-adjoint case and then translate the result back to the case normal case.

A standard result about when integral operators are compact is:

PROPOSITION A.2.4 (Hilbert-Schmidt Operators). *Let (X, μ) and (Y, ν) be measure spaces and let $K : X \times Y \to \mathbf{C}$ be measurable so that*

$$\int_{X \times Y} |K(x, y)|^2 \, d\mu(x) \, d\nu(y) < \infty.$$

*Then the integral operator $T_K f(y) := \int_X K(x, y) f(x) \, d\mu(x)$ is compact as a linear map from $L^2(X) \to L^2(Y)$ and $\|T_K f\|_{L^2} \le \|K\|_{L^2(X \times Y)} \|f\|_{L^2}$. Integral operators with kernels of this form are called **Hilbert-Schmidt operators**.* □

A.3. Miscellaneous analytic facts.

THEOREM A.3.1 (Uniform Boundedness Theorem). *Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. Let $T_\alpha : \mathbf{X} \to \mathbf{Y}$ with $\alpha \in A$ an indexed collection of linear maps. Assume that for each $x \in \mathbf{X}$ that*

$$\sup_{a \in A} \|T_\alpha x\|_{\mathbf{Y}} < \infty.$$

Then there is a constant C so that

$$\|T_\alpha x\|_{\mathbf{Y}} \leq C\|x\|_{\mathbf{X}}$$

for all $\alpha \in A$ and all $x \in \mathbf{X}$. That is there is a uniform upper bound C on the operator norms $\|T_\alpha\|_{\mathrm{Op}}$ of the linear maps T_α.

PROOF. See [**9**, Cor. 21 p. 66] □

Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces and let $B_{\mathbf{X}}$ be the unit ball of $\mathbf{X}$. Then, generalizing a definition above for linear maps between Hilbert spaces, call a linear map $T : \mathbf{X} \to \mathbf{Y}$, a ***compact operator*** iff $T[B_{\mathbf{X}}]$ is precompact in $\mathbf{Y}$. Also recall that a linear map has ***finite rank*** iff the dimension of its image is finite dimensional.

THEOREM A.3.2. *Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces and let* $\mathrm{Compt}(\mathbf{X}, \mathbf{Y})$ *be the set of all compact linear operators from $\mathbf{X}$ to $\mathbf{Y}$. Then*

1. $\mathrm{Compt}(\mathbf{X}, \mathbf{Y})$ *is linear subspace of the space of all bounded linear operators from $\mathbf{X}$ to $\mathbf{Y}$ and is closed with respect to the operator norm $\|\cdot\|_{\mathrm{Op}}$. Thus for any linear T that is a limit (in the operator norm) of compact operators is also compact.*
2. *All finite rank operators form $\mathbf{X}$ to $\mathbf{Y}$ are in* $\mathrm{Compt}(\mathbf{X}, \mathbf{Y})$*. Thus any linear map T form $\mathbf{X}$ to $\mathbf{Y}$ that is a limit (again in the operator norm) of finite rank operators is a compact operator.*

PROOF. This is an instructive exercise. Or see [**9**, §VI.5 pp. 485–487] □

APPENDIX B

Radon Transforms and Spherical Functions on Finite Homogeneous Spaces

B.1. Introduction

In this section we look at the actions of finite groups on finite sets from the point of view of analysis on compact homogeneous and symmetric spaces. As applications we give conditions for some Radon type transforms to be either injective or surjective. Let X be a finite set and let $\ell^2(X)$ be the vector of all real valued functions defined on X. Similar applications hold for Radon transformations on symmetric spaces with actions by Lie groups and at some point I hope to complete the notes above to include some of these results. As a good introduction transforms on homogeneous spaces see Helgason [**17**].

As to the results here for finite group actions I don't think that there is anything new accept maybe the point of view. For more on finite Radon transforms from this viewpoint and the problems treated here see Bloker [**4**], Bolker, Grinberg, and Kung [**5**], Kung [**20**], Grinberg [**15**], Diaconis and Graham [**8**], Frankel and Graham [**12**], and Basterfield and Kelly [**2**]. In [**21**] gives a survey of the finite Radon transform and its applications and in [**26**] surveys the relation between discrete orthogonal polynomials and spherical functions of Chevalley groups with respect to maximal parabolic subgroups. Finally I am told that many of the results here can be treated in a unified method by the use of association schemes. My sources tell me that among the standard sources here are Bannai and Ito [**1**], Biggs [**3**], and Brouwer-Cohen-Neumaier [**6**].

B.2. Finite Homogeneous Spaces

Assume that some finite group G has a transitive action on X then there is the usual permutation representation of G on $\ell^2(X)$ given by $\tau_g f(x) := f(g^{-1}x)$. Fix a point a point $\mathbf{o} \in X$ and let $K := \{a \in G : a\mathbf{o} = \mathbf{o}\}$ be the stabilizer of $\mathbf{o}$. Denote by $\ell^2(X)^K$ the set $\{f \in \ell^2(X) : \tau_a f = f,\ a \in K\}$ of vectors in $\ell^2(X)$ fixed by K. It is clear that the dimension of $\ell^2(X)^K$ is the number of orbits of K acting on X. We call this the ***rank*** of X. (More precisely this should be the rank of the action of G on X.) If $\rho : G \to \mathbf{GL}(V)$ is any representation of G then a linear map $R : \ell^2(X) \to V$ is invariant under G iff $R\tau_g = \rho(g)R$ for all $g \in G$.

Proposition B.2.1. *Let $R : \ell^2(X) \to V$ be invariant. Then R is injective if and only if the restriction $R\big|_{\ell^2(X)^K}$ of R to $\ell^2(X)^K$ is injective.*

Proof. Assume that R is not injective and let $E := \{f : Rf = 0\}$ be the kernel of R. Then as G is transitive on X there is an $f \in E$ with $f(\mathbf{o}) = 1$. As R is invariant the function $p := \frac{1}{|K|}\sum_{a\in K}\tau_a p$ is in E. ($|S|$ is the number of elements in the set S.) Then $p \in \ell^2(X)^K$, $p \neq 0$ (as $p(\mathbf{o}) = 1$) and $Rp = 0$. Thus the restriction of R to $\ell^2(X)^K$ is not injective. The converse is clear. (Note that $\ell^2(X)$ can be replaced by the set of functions $f : X \to \mathbf{F}$ where $\mathbf{F}$ is any field whose characteristic is relatively prime to $|G|$.) □

As an application we consider Radon transforms between finite Grassmannians. Let $\mathbf{F}$ be a finite field and $\mathbf{F}^n$ the vector space of dimension n over $\mathbf{F}$. Then $\mathbf{GL}(\mathbf{F}^n)$ is the group of all invertible linear transformations of $\mathbf{F}^n$ and $\mathbf{Aff}(\mathbf{F}^n)$ is the group of all invertible affine transformations of $\mathbf{F}^n$. We denote by $G_k(\mathbf{F}^n)$ the Grassmannian of all k-dimensional linear subspaces of $\mathbf{F}^n$. (With this notation the n-dimensional projective space over $\mathbf{F}$ is $G_1(\mathbf{F}^{n+1})$.) The affine Grassmannian $AG_k(\mathbf{F}^n)$ is the set all k-dimensional affine subspaces of $\mathbf{F}^n$. For $0 \le k < l \le n-1$ define the ***Radon transform*** $R_{k,l} : \ell^2(AG_k(\mathbf{F}^n)) \to \ell^2(AG_l(\mathbf{F}^n))$ and its dual by $R^*_{k,l} : \ell^2(AG_l(\mathbf{F}^n)) \to \ell^2(AG_k(\mathbf{F}^n))$

$$R_{k,l}f(P) := \sum_{x\subset P} f(x), \qquad R^*_{k,l}F(x) := \sum_{P\supset x} F(P) \tag{B.1}$$

Likewise for $1 \le k < l \le n-1$ there are projective versions of these transforms $P_{k,l} : \ell^2(G_k(\mathbf{F}^n)) \to \ell^2(G_l(\mathbf{F}^n))$ and $P^*_{k,l} : \ell^2(G_l(\mathbf{F}^n)) \to \ell^2(G_k(\mathbf{F}^n))$

$$P_{k,l}f(L) := \sum_{x\subset L} f(x), \qquad P^*_{k,l}F(x) := \sum_{L\subset x} F(L) \tag{B.2}$$

Theorem B.2.2. *Let $0 \le k < l \le n-1$. (a) If $k + l \le n$, then $R_{k,l} : \ell^2(AG_k(\mathbf{F}^n)) \to \ell^2(AG_l(\mathbf{F}^n))$ is injective and the dual map $R^*_{k,l} : \ell^2(AG_l(\mathbf{F}^n)) \to \ell^2(AG_k(\mathbf{F}^n))$ is surjective.*
(b) If $k + l \ge n$ then $R_{k,l} : \ell^2(AG_k(\mathbf{F}^n)) \to \ell^2(AG_l(\mathbf{F}^n))$ is surjective and the dual map $R^*_{k,l} : \ell^2(AG_l(\mathbf{F}^n)) \to \ell^2(AG_k(\mathbf{F}^n))$ is injective.

Theorem B.2.3. *Let $1 \le k < l \le n-1$. (a) If $k + l \le n$, then $P_{k,l} : \ell^2(G_k(\mathbf{F}^n)) \to \ell^2(G_l(\mathbf{F}^n))$ is injective and the dual map $P^*_{k,l} : \ell^2(G_l(\mathbf{F}^n)) \to \ell^2(G_k(\mathbf{F}^n))$ is surjective.*
(b) If $k + l \ge n$ then $P_{k,l} : \ell^2(G_k(\mathbf{F}^n)) \to \ell^2(G_l(\mathbf{F}^n))$ is surjective and the dual map $P^*_{k,l} : \ell^2(G_l(\mathbf{F}^n)) \to \ell^2(G_k(\mathbf{F}^n))$ is injective.

B.3. Injectivity Results for Radon Transforms

The group $\mathbf{GL}(\mathbf{F}^n)$ has a transitive action on $G_k(\mathbf{F}^n)$. Fix $L_0 \in G_k(\mathbf{F}^n)$. Let $K := \{a \in \mathbf{GL}(\mathbf{F}^n) : aL_0 = L_0\}$ be the stabilizer of L_0.

PROPOSITION B.3.1. *The orbits of K on $G_k(\mathbf{F}^n)$ are*

$$X_i = \{L : \dim(L \cap L_0) = i\} \quad \textit{for} \quad \max(0, 2k-n) \le i \le k.$$

Thus the number of orbits of K is $k+1$ for $1 \le k \le n/2$ and $n-k+1$ for $n/2 \le k \le n-1$.

PROOF. Straightforward. □

The affine Grassmannians $AG_k(\mathbf{F}^n)$ are somewhat more complicated. Every $P \in AG_k(\mathbf{F}^n)$ is the translation of some k-dimensional linear subspace of $\mathbf{F}^n$. Let $\mathcal{L}(P) \in G_k(\mathbf{F}^n)$ be the translate of P that contains the origin (and thus is a linear subspace of $\mathbf{F}^n$). Choose $P_0 \in AG_k(\mathbf{F}^n)$ with $0 \in P_0$ (so that $\mathcal{L}(P_0) = P_0$) and let $K := \{a \in \mathbf{Aff}(\mathbf{F}^n) : aP_0 = P_0\}$ be the stabilizer of P_0.

PROPOSITION B.3.2. *The orbits of K on $AG_k(\mathbf{F}^n)$ are*

(B.3)
$$\left.\begin{array}{l} X_{0,i} := \{P : P \cap P_0 = \varnothing,\ \dim(\mathcal{L}(P) \cap P_0) = i\} \\ X_{1,i} := \{P : P \cap P_0 \neq \varnothing,\ \dim(\mathcal{L}(P) \cap P_0) = i\} \end{array}\right\} \textit{for } \max(0, 2k-n) \le i \le k.$$

Thus the number of orbits of K on $AG_k(\mathbf{F}^n)$ is $2(k+1)$ for $0 \le k \le n/2$ and $2(n-k+1)$ for $n/2 \le k \le n-1$.

PROOF. This follows form the last proposition by considering the two cases where $P \cap P_0 = \varnothing$ and $P \cap P_0 \neq \varnothing$. □

Define an inner product $\ell^2(X)$ in the usual manner:

$$\langle f_1, f_2 \rangle := \sum_{x \in X} f_1(x) f_2(x).$$

Then the linear transformations $R_{k,l}$ and $R^*_{k,l}$ are adjoint in the sense that

$$\langle R_{k,l} f, F \rangle = \sum_{P \subset Q} f(P) F(Q) = \langle f, R^*_{k,l} F \rangle.$$

Therefore $R_{k,l}$ is injective if and only if $R^*_{k,l}$ is surjective and $R_{k,l}$ is surjective if and only if $R^*_{k,l}$ is injective. Likewise the maps $P_{k,l}$ and $P^*_{k,l}$ are adjoint.

PROOF OF THEOREM B.2.2. We first prove (a). Thus let $k + l \le n$ and $0 \le k < l \le n-1$. As remarked above the group $G = \mathbf{Aff}(\mathbf{F}^n)$ acts transitively on $AG_k(\mathbf{F}^n)$. Choose $P_0 \in AG_k(\mathbf{F})$ to use as an origin. We assume that $0 \in P_0$ so that $\mathcal{L}(P_0) = P_0$ and let K be the stabilizer of P_0. Let $X_{0,i}$ and $X_{1,i}$ be as in (B.3). Define functions f_i for $0 \le i \le 2k+1$ by

$$f_i(P) := \begin{cases} 1, & 0 \le i \le k \text{ and } P \in X_{0,i}, \\ 1, & k+1 \le i \le 2k+1 \text{ and } P \in X_{1,i-(k+1)}, \\ 0, & \text{otherwise.} \end{cases}$$

Because of the condition $k + l \le n$ it is possible to choose $Q_j \in AG_l(\mathbf{F}^n)$ such that $Q_j \cap P_0 = \varnothing$ and $\dim(\mathcal{L}(Q_j) \cap P_0) = j$ for $0 \le j \le k$ and so that when $k+1 \le j \le 2k+1$ that Q_j contains 0 ($\mathcal{L}(Q_j) = Q_j$) and $\dim(P_0 \cap Q_j) = j - (k+1)$. If $P \in AG_k(\mathbf{F}^n)$, $P \in Q_j$, and $i > j$, then $f_i(P) = 0$. (For example in $k \ge i > j$ then $P \subset Q_j$ implies $P \cap P_0 = \varnothing$ and $\mathcal{L}(P) \cap P_0 \subseteq \mathcal{L}(Q) \cap P_0$ so $\dim(\mathcal{L}(P) \cap P_0) \le \dim(\mathcal{L}(Q_j) \cap P_0) = j < i$. Thus $P \notin X_{0,i}$ so that $f_i(P) = 0$. Similar considerations work in the cases $j \le k < i$ and $k \le j < i$.) This implies $R_{k,l} f_i(Q_j) = 0$ whenever $j < i$. On the other hand when $0 \le i \le k$ we have $c_i := |\{P \subset Q_j : P \in X_{0,i}\}| > 0$ and when $k+1 \le i \le 2k+1$ we also have $c_i := \{P \subset Q_j : P \in X_{1,i-(k+1)}\}| > 0$. There for the matrix $[R_{k,l} f_i(Q_j)]$ is triangular

$$[R_{k,l} f_i(Q_j)] = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ * & c_1 & 0 & \cdots & 0 \\ * & * & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ * & * & * & \cdots & c_{2k+1} \end{bmatrix}$$

and as the $c_i \ne 0$ this matrix is nonsingular. But then the functions $R_{k,l} f_i$, $i = 0, \dots, 2k-1$ are linearly independent (if $\sum_{i=0}^{2k+1} a_i f_i = 0$, then by evaluating at the Q_j's we get a nonsingular system for the a_i's.) As the functions $f_0, \dots, f_{2k+1}$ are a basis of $\ell^2(\mathbf{F}^n)^K$ this implies the restriction of $R_{k,l}$ to $\ell^2(\mathbf{F}^n)^K$ is injective. Thus $R_{k,l}$ is injective, and $R^*_{k,l}$ surjective when $k + l \le 0$.

We now assume $0 \le k < l \le n-1$ and $k + l \ge n$ and show $R^*_{k,l}$ is injective. These conditions imply $l \ge n/2$. Let $Q_0 \in AG_l(\mathbf{F}^n)$ be so that $0 \in Q_0$ (and thus $\mathcal{L}(Q_0) = Q_0$) and let $K := \{a \in \mathbf{Aff}(\mathbf{F}^n) : aQ_0 = Q_0\}$ be the stabilizer of Q_0. Then $l \le n/2$ implies K has $(2l - n + 1)$ orbits on $AG_l(\mathbf{F}^n)$. To simplify notation let $r = 2n - l$ be the codimension of Q_0. Then proposition B.3.2 implies the orbits of K are

$$\begin{array}{l} Y_{0,i} := \{Q : Q \cap Q_0 \ne \varnothing,\ \dim(\mathcal{L}(Q) + Q_0) = l + i\} \\ Y_{1,i} := \{Q : Q \cap Q_0 = \varnothing,\ \dim(\mathcal{L}(Q) + Q_0) = l + (i - r - 1)\} \end{array} \quad \text{for } 0 \le i \le r.$$

Define functions F_i on $AG_l(\mathbf{F}^n)$ by

$$F_i(Q) := \begin{cases} 1, & 0 \le i \le r \text{ and } Q \in Y_{0,i}, \\ 1, & r+1 \le i \le 2r+1 \text{ and } P \in Y_{1,i-(r+1)}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $F_0, \dots, F_{2k+1}$ is a basis of the isotropic functions $\ell^2(AG_l(\mathbf{F}^n))^K$. Because of the dimension restriction $k + l \ge n$ it is possible to choose elements $P_j \in AG_k(\mathbf{F}^n)$ so that $P_j \cap Q_0 \ne \varnothing$, $\dim(\mathcal{L}(P_j) + Q_0) = l + j$ for $0 \le j \le r$ and $P_j \cap Q_0 = \varnothing$, $\dim(\mathcal{L}(P_j) + Q_0) = l + (j - r - 1)$ for $r + 1 \le j \le 2r + 1$. But then by considering the cases $0 \le i < j \le r$, $0 \le i \le r < j \le 2r + 1$ and $r + 1 \le i < j \le 2r + 1$ it follows that if $i < j$ and $Q \supset P_j$, then $F_i(Q) = 0$. Thus $i < j$ implies $R^*_{k,l} F_i(P_j) = 0$. But clear $R^*_{k,l} F_i(P_i) \ne 0$. Whence $[R^*_{k,l} F_i(Q_j)]$ is a triangular matrix with non-zero elements along the diagonal and thus is nonsingular. Just as in the last case this implies that $R_{k,l}^*F_0, \dots, R_{k,l}^*R_{2r+1}$ are independent which in turn implies the restriction of $R_{k,l}^*$ to the isotropic functions $\ell^2(\mathbf{F}^n)^K$ is injective which by Proposition B.2.1 implies $R_{k,l}^*$ is injective. Then $R_{k,l}$ is surjective by duality. $\square$

PROOF OF THEOREM B.2.3. An easy variant on the last proof. $\square$

B.4. The Convolution Algebra of a Finite G-Space

Let X be a finite set. Let $\ell^2(X \times X)$ be the set of real valued function $h : X \times X \to \mathbf{R}$. Then for each $h \in \ell^2(X \times X)$ define a linear map $T_h : \ell^2(X) \to \ell^2(X)$ by

$$T_h f(x) := \sum_{y \in X} h(x, y) f(y).$$

If $f \in \ell^2(X)$ is viewed as a column vector with entries indexed by X (in some ordering) and h as a matrix with entries indexed by $X \times X$ then the linear operator T_h is just matrix multiplication by h. Define the natural product (corresponding to matrix multiplication) on $\ell^2(X \times X)$ by

$$h * k(x, y) := \sum_{z \in X} h(x, z) k(z, y).$$

Then $T_h \circ T_k = T_{h*k}$ as expected. If $h \in \ell^2(X \times X)$ then let $h^t(x, y) = h(y, x)$ be the "transpose" of h. Then the linear operator T_{h^t} is the adjoint of T_h in the sense that

$$\langle T_h f_1, f_2 \rangle = \langle f_1, T_{h^t} f_2 \rangle$$

so that T_h is selfadjoint if and only if h is symmetric $h(x, y) = h(y, x)$.

Let G be a finite group and assume that X is a G-space, that is the group G has an action on X on the left $(g, x) \mapsto gx$. Then there is the usual permutation representation $\tau : G \to O(\ell^2(X))$ (here $O(\ell^2(X))$ is the orthogonal group of $\ell^2(X)$) given by

$$\tau_g f(x) := f(g^{-1}x).$$

The group G then has the action $g(x, y) = (gx, gy)$ on $X \times X$. Let $\ell^2(X \times X)^G$ be the subspace of $\ell^2(X \times X)$ of functions invariant under G. That is $h \in \ell^2(X \times X)$ if and only if $h(gx, gy) = h(x, y)$. These definitions imply:

PROPOSITION B.4.1. *If $h \in \ell^2(X \times X)^G$ then the linear operator T_h commutes with the action of G, that is $T_h \tau_g = \tau_g T_h$ for all $g \in G$. The set $\ell^2(X \times X)^G$ is closed under the product $*$, $h * k \in \ell^2(X \times X)^G$ if $h, k \in \ell^2(X \times X)^G$.* $\square$

This implies that with the product $*$ the vector space $\ell^2(X \times X)^G$ becomes an algebra with identity (the function $\delta(x, y) = 1$ for $x = y$ and $= 0$ for $x \neq y$ is the identity). Because of analogues form functional analysis we call $\ell^2(X \times X)^G$ the ***convolution algebra*** of X. This is of most interest when the action of G is transitive on X. In this case choose a point $\mathbf{o} \in X$ to use as an origin and let

$$K := \{a \in G : a\mathbf{o} = \mathbf{o}\}$$

be the stabilizer of $\mathbf{o}$. In this case let $\ell^2(X)^K$ be the set of elements in $\ell^2(X)$ invariant under K, that is $\ell^2(X)^K = \{f : f(ax) = f(x) \text{ for all } a \in K\}$. Functions in $\ell^2(X)^K$ will be called ***isotropic*** or ***radial***. It is clear that the dimension of $\ell^2(X)^K$ is equal to the number of orbits of K acting on X. This is also the dimension of $\ell^2(X \times X)^K$ because of:

Remark B.4.2. If the action of G is transitive on X then there is a linear isomorphism $\mathcal{R} : \ell^2(X \times X)^G \to \ell^2(X)^K$ given by $\mathcal{R}h(y) := h(\mathbf{o}, y)$. The inverse of $\mathcal{R}$ is given by

$$h(x,y) = \mathcal{R}^{-1}f(x,y) = f(\xi^{-1}y) \qquad \text{where } \xi\mathbf{o} = x. \tag{B.4}$$

(This is independent of the choice of ξ with $\xi\mathbf{o} = x$.) Thus

$$\dim \ell^2(X \times X)^G = \dim \ell^2(X)^K = \text{number of orbits of } K \text{ on } X$$

We now would like to give a standard basis of $\ell^2(X \times X)^G$. Let r be the rank of the action of G on X. That is the stabilizer K of $\mathbf{o}$ has r orbits $X_1, \dots, X_r$ and we assume that $X_1 = \{\mathbf{o}\}$. Define $e_i \in \ell^2(X \times X)$ by

$$e_k(x,y) := \begin{cases} 1, & \xi^{-1}y \in X_k \\ 0, & \xi^{-1}y \notin X_k \end{cases} \qquad \text{where } \xi\mathbf{o} = x.$$

It is easily checked this is defined independently of the choice of ξ with $\xi\mathbf{o} = x$ and that $e_k(gx, gy) = e_k(x, y)$. These are clearly linearly independent and thus form a basis of $\ell^2(X \times X)^G$. Let

$$f_k(y) := e_k(\mathbf{o}, y) = \begin{cases} 1, & y \in X_k \\ 0, & y \notin X_k \end{cases}$$

be the corresponding functions in $\ell^2(X)^K$ and L_k the linear operator

$$L_k f(x) = \sum_{y \in X} e_k(x,y) f(y). \tag{B.5}$$

These linear operators have a combinatorial interpretation. For $k = 1, \dots, r$ define a directed graph $\mathcal{G}_k$ with vertices X and so that there is an edge point from x to y iff $e_k(x, y) = 1$. Then the matrix $[e_k(x, y)]_{x,y \in X}$ is just the incidence matrix of the graph $\mathcal{G}_k$. (The linear operator $L_k - c\,\mathrm{Id}$ where $c := |\{x : e_k(\mathbf{o}, x) \neq 0\}|$ is often called the ***Laplacian*** of $\mathcal{G}_k$). The operator L_k is somewhat analogous to a differential operator $f \mapsto Df$, where $Df(x)$ is computed in terms of the points y "infinitely close" to x, for if $f \in \ell^2(X)$, then

$$L_i f(x) = \sum_{y \in X} e_k(x,y) f(y) = \sum_{y \text{ connected to } x \text{ in } \mathcal{G}_k} f(y)$$

and thus computing $L_k f(x)$ only involves nearest neighbors of x in $\mathcal{G}_k$.

For each $k = 1, \dots, r$ choose $x_k \in X_k$ to use as a reference point. Then we define some numbers related to the combinatorics of the action of G on X, or more precisely to the combinatorics of the graphs $\mathcal{G}_k$. Let

$$n_k = |X_k| \tag{B.6}$$

and

$$\begin{aligned} l_{ij}^{(k)} &= \text{number of points in } X_j \text{ connected to } x_i \text{ in } \mathcal{G}_k \\ &= |\{y \in X_j : e_k(x_i, y) = 1\}| = \sum_{y \in X} e_k(x_i, y) f_j(y) \\ &= L_k f_j(x_i). \end{aligned} \tag{B.7}$$

This definition is independent of the choice of the reference point $x_i \in X_i$. The functions $f_1, \dots, f_r$ are basis of $\ell^2(X)^K$ and each L_k maps $\ell^2(X)^K$ into itself. In this basis the inner product is given by

$$\langle f_i, f_j \rangle = \delta_{ij} n_i \tag{B.8}$$

and the linear operators L_k satisfy

$$L_k f_i = \sum_{j=1}^{r} l_{ji}^{(k)} f_j. \tag{B.9}$$

This follows easily form B.7. Thus in this basis the matrix of L_k viewed as a linear map $L_k : \ell^2(X)^K \to \ell^2(X)^K$ is

$$[L_k] = \begin{bmatrix} l_{11}^{(k)} & l_{12}^{(k)} & \cdots & l_{1r}^{(k)} \\ l_{21}^{(k)} & l_{22}^{(k)} & \cdots & l_{2r}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ l_{r1}^{(k)} & l_{2r}^{(k)} & \cdots & l_{rr}^{(k)} \end{bmatrix} \tag{B.10}$$

While we will not use this fact here it is worth pointing out that when G is transitive on X that the convolution algebra of X is isomorphic to a subalgebra of the group ring of G. Let $\mathbf{R}[G]$ be the group ring of G viewed as functions $f : G \to \mathbf{R}$ and let $\mathbf{R}[G]^{K \times K}$ be the functions that are bi-invariant under K, that is $f(a\xi b) = f(\xi)$ for all $a, b \in K$. Then $\mathbf{R}[G]^{K \times K}$ is a sub-ring of $\mathbf{R}[G]$ isomorphic to $\ell^2(X \times X)^G$ with the product $*$. (See Exercise 3.2.8 page 32). This is the motivation for calling $\ell^2(X \times X)$ with the product $*$ the convolution algebra, as in the case of locally compact groups with the Haar measure the analogue of the group algebra is $L^1(G)$ with the convolution product $f_1 * f_2(\xi) = \int_G f_1(\xi\eta^{-1})(\eta)\, d\eta$.

B.5. Finite Symmetric Spaces

Let X be a finite set and G a finite group acting on X. Call the action of G on X ***symmetric*** if and only if for each pair $x, y \in X$ there is a $g \in G$ that interchanges x and y (i.e. $gx = y$ and $gy = x$). Note that if the action of G is symmetric on X, then it is transitive on X.

Proposition B.5.1. *If the action of G on X is symmetric then every $h \in \ell^2(X \times X)^G$ is symmetric $h(x,y) = h(y,x)$ and the product $*$ is commutative. Therefore the set of linear maps $\{T_h : h \in \ell^2(X \times X)^K\}$ is a commuting set of self-adjoint linear operators.*

Proof. If $x, y \in X$ then there is a $g \in G$ that interchanges them. Thus by the basic invariance property of elements of $\ell^2(X \times X)^K$, $h(x,y) = h(gx, gy) = h(y,x)$. If $h, k \in \ell^2(X \times X)^G$, then using the symmetry of h, k and $h * k$

$$\begin{aligned} h * k(x,y) &= \sum_{z \in X} h(x,z)k(z,y) = \sum_{z \in X} k(y,z)h(z,x) \\ &= k * h(y,x) = k * h(x,y). \end{aligned}$$

Which shows $*$ is commutative. □

We now fix some notation. As above we choose an origin $\mathbf{o} \in X$ and let K be the stabilizer of $\mathbf{o}$ in G. If E is a G invariant subspace of $\ell^2(X)$ then let

$$E^K := \{f \in E : \tau_a f = f \text{ for all } a \in K\}$$

be the isotropic elements of E. As $\{T_h : h \in \ell^2(X \times X)^G\}$ is a commuting set of selfadjoint linear maps then they can be simultaneously diagonalized. Put somewhat differently this means there is a finite set of nonzero linear functionals $\alpha_1, \dots, \alpha_r : \ell^2(X \times X)^G \to \mathbf{R}$ (called ***weights*** so that the corresponding ***weight space***

$$E_{\alpha_i} := \{f \in \ell^2(X) : T_h f = \alpha_i(h) f \text{ for all } h \in \ell^2(X \times X)^G\}$$

is not the zero subspace $\{0\}$. Then the spectral theorem implies

$$\ell^2(X) = E_0 \oplus \bigoplus_{i=1}^{r} E_{\alpha_i} \qquad \text{(orthogonal direct sum)}, \tag{B.11}$$

where $E_0 := \{f : T_h f = 0 \text{ for all } h \in \ell^2(X \times X)^G\}$.

Theorem B.5.2. *Let X have a symmetric action by the group G. Then*

1. $\ell^2(X) = \bigoplus_{i=1}^{r} E_{\alpha_i}$ *(orthogonal direct sum)*
2. *Each E_{α_i} is irreducible.*
3. *Each $E_{\alpha_i}^K$ is one dimensional and spanned by a unique element p_{α_i} with $p_{\alpha_i}(\mathbf{o}) = 1$ called the **spherical function** in $E_{\alpha_i}^K$.*
4. *If $i \neq j$ then E_{α_i} and E_{α_j} are not isomorphic as G-modules.*
5. *If E is any irreducible G invariant subspace, then $E = E_{\alpha_i}$ for some i.*
6. *r = number of orbits of K on X = $\dim \ell^2(X)^K = \dim \ell^2(X \times X)^G$. That is r is the rank of the action of G on X.*

PROOF. Let $f \in E_0$ and let $\delta \in \ell^2(X \times X)^G$ be the identity matrix. Then $f = T_\delta f = 0$ so $E_0 = \{0\}$. Using this in equation (B.11) shows part 1 holds.

If $E \subseteq \ell^2(X)$ is any nonzero G invariant subspace there is some element $f \in E$ with $f(\mathbf{o}) = 1$. Then the element $p(x) = \frac{1}{|K|}\sum_{a\in K} f(ax)$ is in E^K. This shows that E^K has an element p with $p(\mathbf{o}) = 1$. We now claim that if $f \in E^K_{\alpha_i}$ and $f(\mathbf{o}) = 0$ then $f \equiv 0$. To see define, as in remark B.4.2, a function $h \in \ell^2(X \times X)$ by equation (B.4). Then $h(\mathbf{o}, y) = f(y)$ and

$$0 = \alpha_i(h)f(\mathbf{o}) = T_h f(\mathbf{o}) + \sum_{y\in X} h(\mathbf{o}, y)f(y) = \sum_{y\in X} f(x)^2$$

which shows that $f \equiv 0$ as claimed.

The arguments in the last paragraph imply that $E^K_{\alpha_i}$ is one dimensional and is spanned by a unique element p_{α_i} with $p_{\alpha_i}(\mathbf{o}) = 1$. This in turn implies E_{α_i} is irreducible as if not it could be decomposed as a direct sum $E_{\alpha_i} = F_1 \oplus F_2$ and thus $E^K_{\alpha_i} = E^K_1 \oplus E^K_2$ and each F^K_1 is at least one dimensional, contradicting that $E^K_{\alpha_i}$ is one dimensional. This proves parts 2 and 3.

LEMMA B.5.3. *Let $f_1, f_2 \in \ell^2(X)$. Then there is a constant c_{α_i} so that for all $f \in E_{\alpha_i}$*

$$\sum_{g\in G, y\in X} f_1(g^{-1}x)f_2(g^{-1}y)f(y) = c_{\alpha_i}(f_1, f_2)f(x).$$

PROOF. Let $h(x, y) = \sum_{g\in G} f_1(g^{-1}x)f_2(g^{-1}y)$. Then for any $\xi \in G$ a change of sum in the sum defining h implies $h(\xi x, \xi y) = h(x, y)$. Thus $h \in \ell^2(X \times X)^G$ and so for any f in E_{α_i}, $T_h f = \alpha_i(h)f$. This is equivalent to the statement of the lemma with $c_{\alpha_i}(f_1, f_2) = \alpha_i(h)$. □

We now show that if $\alpha, \beta \in \{\alpha_1, \dots, \alpha_r\}$ then E_α and E_β are not isomorphic. Let $\chi_\alpha(g) := \mathrm{trace}(\tau_g|_{E_\alpha})$ and $\chi_\beta(g) := \mathrm{trace}(\tau_g|_{E_\beta})$ be the characters of the representation restricted to E_α and E_β. Then $f_{1\alpha}, \dots, f_{l\alpha}$ be an orthonormal basis of E_α and $f_{1\beta}, \dots, f_{m\beta}$ an orthogonal basis of E_β. Then the matrix of $\tau|_{E_\alpha}$ is $[\langle f_{i\alpha}, \tau_g f_{j\alpha}\rangle]$. The trace is the sum of the diagonal elements so

$$\begin{aligned}
\sum_{g\in G}\chi_\alpha(g)\chi_\beta(g) &= \sum_{g\in G}\sum_{i,j}\langle f_{i\alpha},\tau_g f_{i\alpha}\rangle\langle f_{j\beta},\tau_g f_{j\beta}\rangle \\
&= \sum_{i,j}\sum_{x\in X}\left(\sum_{y\in X}\sum_{g\in G} f_{i\alpha}(g^{-1}x)f_{j\beta}(g^{-1}y)f_{i\alpha}(y)\right) f_{j\beta}(x) \\
&= \sum_{i,j} c_\alpha(f_{i\alpha},f_{j\beta})\sum_{x\in X} f_{i\alpha}(x)f_{j\beta}(x) \qquad \text{(by the lemma)} \\
&= \sum_{i,j} c_\alpha(f_{i\alpha},f_{j\beta})\langle f_{i\alpha},f_{j\beta}\rangle \\
&= 0
\end{aligned}$$

as E_α is orthogonal to E_β. But is E_α and E_β are isomorphic then $\chi_\alpha = \chi_\beta$ which would lead to the contradiction $0 = \sum_{g\in G}\chi_\alpha(g)^2 > 0$. This proves part 4. The last two parts follow from Schur's lemma and easy linear algebra. □

B.6. Invariant Linear Operators on Finite Symmetric Spaces

We use the notation of the last section. That is G has a symmetric action on the set X and we use the notation of Theorem B.5.2.

THEOREM B.6.1. *Let X have a symmetric action by G and let $L : \ell^2(X) \to \ell^2(X)$ an invariant linear operator (that is $L\tau_g = \tau_g L$ for all $g \in G$). Then for each i there holds $LE_{\alpha_i} \subseteq E_{\alpha_i}$ and $L|_{E_{\alpha_i}}$ is multiplication*

$$L|_{E_{\alpha_i}} = c_i \operatorname{Id}_{E_{\alpha_i}} \quad \textit{where} \quad c_i = (Lp_{\alpha_i})(\mathbf{o}).$$

In particular L is invertible if and only if $(Lp_{\alpha_i})(\mathbf{o}) \neq 0$ for all i. In this case the inverse is given by

$$L^{-1} = \sum_{i=1}^{r} \frac{1}{Lp_{\alpha_i}(\mathbf{o})}\pi_i$$

where $\pi_i : \ell^2(X) \to E_{\alpha_i}$ is orthogonal projection.

PROOF. This follows form Theorem B.5.2 and Schur's lemma. ($c_i = (Lp_{\alpha_i})(\mathbf{o})$ because $p_{\alpha_i}(\mathbf{o}) = 1$.) □

Let $\rho : G \to V$ be a representation of G on a real vector space V. Then a linear operator $L : \ell^2(X) \to V$ is invariant iff $L\tau_g = \rho(g)L$ for all $g \in G$.

THEOREM B.6.2. *For an invariant linear operator $L : \ell^2(X) \to V$ the following are equivalent:*

1. *L is injective*
2. *$Lp_{\alpha_i} \neq 0$ for all i.*
3. *The restriction $L|_{\ell^2(X)^K}$ of L to the isotropic functions $\ell^2(X)^K$ is injective.*

If L is injective it is inverted by

$$f = \left(\sum_{i=1}^{r} \frac{1}{L^* L p_{\alpha_i}(\mathbf{o})} \pi_i L^* \right) Lf$$

PROOF. The equivalence of the three conditions follows form Proposition B.2.1 and Theorem B.5.2. The formula $\langle L^* Lf, f \rangle = \langle Lf, Lf \rangle$ shows that $L : \ell^2(X) \to V$ is injective if and only if $L^* L : \ell^2(X) \to \ell^2(X)$ is injective. The inversion formula now follows from the last theorem. □

For these results to be of interest in concrete cases it is clear that methods for finding the spherical functions are needed. The following gives one method.

PROPOSITION B.6.3. *A function $p \in \ell^2(X)^K$ is a spherical function if and only if it is a joint eigenfunction of the operators L_k defined by equation (B.5) and $p(\mathbf{o}) = 1$. If for some k the restriction of L_k to $\ell^2(X)^K$ has r distinct eigenvalues, then any eigenfunction p of $L_k|\ell^2(X)^K$ satisfying $p(\mathbf{o}) = 1$ is a spherical function.*

PROOF. The functions e_k with $k = 1, \ldots, r$ are a basis for $\ell^2(X \times X)^G$ and thus any function that is a joint eigenfunction for the $L_k = T_{e_k}$ is a joint eigen function for all the linear operators T_h, $h \in \ell^2(X \times X)^G$. Thus the joint eigenspaces of the L_k's are just the E_{α_i}'s. As each E_{α_i} contains a unique isotropic function the first part follows. If L_k has r distinct eigenvalues, then the r linear operators $I, L_k, L_k^2, \ldots, L_k^{r-1}$ are linearly independent and therefore they span the set $\{T_h : h \in \ell^2(X \times X)^K\}$. Thus the eigenspaces of L_k are the same as the joint eigenspaces of $\{T_h : h \in \ell^2(X \times X)^K\}$ □

Where the action of G on X is symmetric then for each k $e_k(x, y) = e_k(y, x)$ which implies the linear laps L_k are self-adjoint. Using the form of the inner product in the basis $f_1, \ldots, f_r$ of $\ell^2(X)^K$ the relation $\langle L_k f_i, f_j \rangle = \langle f_i, L_k f_j \rangle$ becomes

$$n_i l_{ij}^{(k)} = n_j l_{ji}^{(k)} \tag{B.12}$$

We also note that knowing the spherical functions p_{α_i} allows one to write down the matrix for the orthogonal projections $\pi_i : \ell^2(X) \to E_{\alpha_i}$. Let $h_i \in \ell^2(X \times X)^G$ be the unique element so that

$$h_i(\mathbf{o}, y) = p_{\alpha_i}(y).$$

Then

$$T_{h_i} p_{\alpha_j} = \sum_{y \in X} h_i(\mathbf{o}, y) p_{\alpha_j}(y) = \sum_{y \in X} p_{\alpha_i}(y) p_{\alpha_j}(y) = \|p_{\alpha_i}\|^2 \delta_{ij}.$$

Thus the projection onto E_{α_i} is

$$\pi_i = \frac{1}{\|p_{\alpha_i}\|^2} T_{h_i}.$$

B.7. Radon Transforms for Doubly Transitive Actions

The action of G on X is doubly transitive iff for any two ordered ordered pairs $(x_1, y_1), (x_2, y_2) \in X \times X$ with $x_i \neq y_i$ for $i = 1, 2$ there is an element $g \in G$ with $gx_1 = x_2$ and $gy_1 = y_2$. Such an action is clearly symmetric. As before fix $\mathbf{o} \in X$ to use as an origin and let $K = \{a \in G : a\mathbf{o} = \mathbf{o}\}$ be the stabilizer of $\mathbf{o}$. Then G is double transitive if and only if K has exactly two orbits on X, the one element orbit $X_1 = \{\mathbf{o}\}$ and the orbit $X_2 := X \setminus \{\mathbf{o}\}$. This means that G in the decomposition of theorem B.5.2 that $l = 2$ so that $\ell^2(X) = E_1 \oplus E_2$. As the space of constant functions and and space of functions that sum to zero are both invariant under G we see

$$E_1 := \text{The constant functions}, \qquad E_2 := \left\{ f : \sum_{x \in X} f(x) = 0 \right\}.$$

Then the spherical functions are

$$p_1(x) \equiv 1, \qquad p_2(x) = \begin{cases} 1, & x = \mathbf{o} \\ \dfrac{-1}{|X| - 1}, & x \neq \mathbf{o} \end{cases}$$

The orthogonal projections of $\ell^2(X)$ onto these E_1 and E_2 are given by

$$\pi_1(f)(x) \equiv \frac{1}{|X|} \sum_{y \in X} f(y), \qquad \pi_2 f(x) = f(x) - \frac{1}{|X|} \sum_{y \in X} f(y)$$

Let L_0 be any nonempty subset of X other than X its self and let

$$\overline{X} = \{gL_0 : g \in G\}.$$

be the set of G translates of L_0. If $\overline{K} := \{g \in G : gL_0 = L_0\}$ then $|\overline{X}| = |G|/|\overline{K}|$. There is a natural Radon transform $R : \ell^2(X) \to \ell^2(\overline{X})$ given by

$$Rf(L) := \sum_{x \in L} f(x).$$

There is a dual transform $R^* : \ell^2(\overline{X}) \to \ell^2(X)$

$$R^* F(x) := \sum_{L \ni x} F(L).$$

We note that R^* is the adjoint of R in the sense that

$$\langle Rf, F \rangle_{\ell^2(\overline{X})} = \sum_{x \in L} f(x) F(L) = \langle f, R^* F \rangle_{\ell^2(X)} \tag{B.13}$$

Therefore the map R is injective if and only if the map R^* is surjective.

The image of the spherical functions p_1 and p_2 under R is

$$Rp_1(L) \equiv |L_0|, \qquad Rp_2(L) = \begin{cases} \dfrac{|X| - |L_0|}{|X| - 1}, & \mathbf{o} \in L \\[2ex] \dfrac{-|L_0|}{|X| - 1}, & \mathbf{o} \notin L \end{cases}$$

For $x \in X$ let $m = |\{L \in \overline{X} : x \in L\}|$ be the number of elements of $\overline{X}$ that contain x (this is independent of x). Then by counting the pairs (x, L) with $x \in L$ in two ways (first summing on x and then on L, or first summing on L and then on x)

$$m = \frac{|L_0||\overline{X}|}{|X|}. \tag{B.14}$$

Then the images of Rp_1 and Rp_2 under R^* and evaluated at $\mathbf{o}$ are

$$R^*Rp_1(\mathbf{o}) = m|L_0|, \qquad R^*Rp_2(\mathbf{o}) = m\frac{|X| - |L_0|}{|X|}.$$

The operator R^*R is G invariant thus the results of the last section lead to

THEOREM B.7.1. *If the action of G on X is doubly transitive then the Radon transform $R : \ell^2(X) \to \ell^2(\overline{X})$ is injective and is inverted by*

$$f = \frac{1}{m}\left(\frac{1}{|L_0|}\pi_1 R^* + \frac{|X|}{|X| - |L_0|}\pi_2 R^*\right) Rf$$

where m, π_1 and π_2 are as above. Thus $|X| \le |\overline{X}|$. By duality the transform $R^* : \ell^2(\overline{X}) \to \ell^2(X)$ is surjective. □

By applying this to the characteristic functions of sets $A, B \subseteq X$:

COROLLARY B.7.2. *If G has a doubly transitive action on X and with $A, B, L_0 \subset X$ and $L_0 \neq \varnothing, X$ and $|A \cap gL_0| = |B \cap gL_0|$ for all $g \in G$ then $A = B$.* □

For any finite field the action of $\mathbf{Aff}(\mathbf{F}^n)$ is doubly transitive on $\mathbf{F}^n$ if $x \in \mathbf{F}^n = AG_0(\mathbf{F}^n)$, then the set of $P \in AG_l(\mathbf{F}^n)$ is isomorphic to $G_1(\mathbf{F}^{n-1})$ thus the above specializes to

COROLLARY B.7.3. *The radon transform $R_{0,l} : \ell^2(\mathbf{F}^n) \to \ell^2(AG_l(\mathbf{F}^n))$ is injective and inverted by*

$$f = \frac{1}{|G_1(\mathbf{F}^n)|}\left(\frac{1}{|\mathbf{F}^l|}\pi_1 R^*_{0,l} + \frac{|\mathbf{F}^n|}{|\mathbf{F}^n| - |\mathbf{F}^l|}\pi_2 R^*_{0,l}\right) R_{0,l} f.$$

There is a corresponding result in the projective case:

COROLLARY B.7.4. *The radon transform $P_{1,l} : \ell^2(G_1(\mathbf{F}^n)) \to \ell^2(G_l(\mathbf{F}^n))$ is injective and inverted by*

$$f = \frac{1}{|G_{l-1}(\mathbf{F}^{n-1})|}\left(\frac{1}{|G_1(\mathbf{F}^l)|}\pi_1 P^*_{1,l} + \frac{|G_1(\mathbf{F}^n)|}{|G_1(\mathbf{F}^n)| - |G_1(\mathbf{F}^l)|}\pi_2 P^*_{1,l}\right) P_{1,l} f.$$

APPENDIX C

Fiber Integral and the Coarea Formula

C.1. The basic geometry of the fibers of a smooth map

Our first goal is to understand when a fiber of a smooth map is a smooth manifold. The basic tools here are Sard's theorem and the implicit function function theorem. We start by fixing our notation and giving the basic definitions.

For a smooth manifold M^m (superscripts denote dimension) the tangent bundle of M will be denoted by $T(M)$ and the tangent space to M at x will be is $T(M)_x$. If $f : M^m \to N^n$ is a smooth map then the derivative map form $T(M)_x$ to $T(N)_{f(x)}$ will be denoted be f_{*x} or df_x. If $X \in T(M)_x$ is a tangent vector, then the image of X under the derivative of f is denoted by $f_{*x}X$ or $df_x(X)$. Often this will be shortened to f_*X or $df(X)$.

If $f : M^m \to N^n$ is smooth and $\in M$, then f_{*x} has ***full rank*** if and only if $\operatorname{rank}(f_{*x}) = \min(m, n)$. The function f is said to have full rank at x if and only if f_{*x} has full rank. Thus if $m \leq n$ the map f has full rank at x if and only if f_{*x} is injective and if $m \geq n$ it has full rank at x iff and only if f_{*x} is surjective. A point where f has full rank is called a ***regular point*** of f. Any any other point is called a ***critical point*** of f. Therefore x is a critical point of $f : M^m \to N^n$ if and only if $\operatorname{rank}(f_{*x}) < \min(m, n)$. A point $y \in N^n$ is a ***critical value*** of $f : M^m \to N^n$ if and only if $y = f(x)$ for some critical point x of f. A point $y \in N^n$ is a ***regular value*** of f if and only if it is not a critical value of f. Therefore y is a regular value of f if and only if every point of the fiber $f^{-1}[y]$ is a regular point of f. Note this includes the case when $f^{-1}[y]$ is empty (this a point $y \in N^n$ that is not a value of f still manages to be a critical value of f). The fibers over regular values are very well behaved as the following shows:

THEOREM C.1.1 (Geometric Implicit Function Theorem). *If $m \geq n$ and $f : M^m \to N^n$ is a smooth map, then for every regular point y the preimage $f^{-1}[y]$ is a smooth imbedded submanifold of M^m of dimension $m - n$. This includes the case when $m = n$ (where a zero dimensional submanifold is discrete subset of M^m) and the case where $f^{-1}[y]$ is empty (so by convention the empty set is a submanifold of any dimension we please.)*

EXERCISE C.1.2. If this version of the implicit function is new to you, then use what ever version of it you are used to and prove the geometric version. Then use the geometric version to prove your standard version. □

Our next goal is Sard's theorem which says that almost every point in N^n is a regular value of $f : M^m \to N^n$ is a regular point of f. We start by giving the definition of the sets of measure zero on smooth manifolds. Let M^m be a smooth manifold and let $\{(U_\alpha, x^1_\alpha, \dots, x^m_\alpha)\}_{\alpha \in A}$ be a countable cover of M^m by coordinate charts $(U_\alpha, x^1_\alpha, \dots, x^m_\alpha)$. In each coordinate domain U_α there is the Lebesgue measure μ_α defined by the coordinate functions $x^1_\alpha, \dots, x^m_\alpha$. That is $\mu_\alpha := dx^1_\alpha \cdots dx^m_\alpha$. A set $S \subseteq M^m$ is said to have ***measure zero*** if and only if $\mu_\alpha(U_\alpha \cap S) = 0$ for all α. A set $P \subseteq M^m$ is has ***full measure*** if and only if it is the complement in M of a set of measure zero. A property is said to hold ***almost everywhere*** on M^m if and only if the set of points where the property holds is a set of full measure.

Exercise C.1.3. Show these definitions are independent of the choice of the coordinate cover $\{(U_\alpha, x^1_\alpha, \dots, x^m_\alpha)\}_{\alpha \in A}$. That is if $\{(U_\beta, x^1_\beta, \dots, x^m_\beta)\}_{\beta \in B}$ is anther countable set of coordinate charts covering M^m, then this leads to the same collection of sets of measure zero, and thus the same notation of almost everywhere. Also show that a countable union of sets of measure zero is a set of measure zero. □

Exercise C.1.4. Let g be a Riemannian metric on M^n. (In a local coordinate system $g = \sum g_{ij} dx^i dx^j$). Then g determines the usual Riemannian volume measure μ_g (in local coordinates $\mu_g = \sqrt{\det(g_{ij})} dx^1 \cdots dx^m$). Show that $S \subset M^m$ has measure zero if and only if $\mu_g(S) = 0$. (This also shows that all Riemannian metrics on M determine the same sets of measure zero.) □

Theorem C.1.5 (Sard's Theorem). *If $f : M^m \to N^n$ is a smooth map, then the set of critical values of f has measure zero in N^n. Thus if $m \geq n$ by the geometric implicit function theorem $f^{-1}[y]$ is a smooth imbedded submanifold of M^m of dimension $m - n$ for almost all $y \in N^n$.*

Remark C.1.6. This result was first given by Sard [**24**] in 1948 who shows the result is true under the weaker smooth assumption that f is of class C^k where $k \geq \max\{1, m - n + 1\}$. A proof of this can be found in [**27**, p. 47]. The bound on k is sharp as is seen from a famous example of Whitney [**30**]. □

Proof. The proof is by induction on $m = \dim(M)$. If $m < n$ the result is not hard and left to the reader . In fact in the case $m < n$ it is true that $f[M^m]$ has measure zero in N^n (see the exercise following the proof).

We next note that the result for manifolds for maps $f : \mathbf{R}^m \to \mathbf{R}^n$ implies the result for maps between manifolds. To see this let $f : M^m \to N^n$. Then it is possible to choose a countable set collection of coordinate charts $\{(U_\alpha, x^i_\alpha)\}$ on M^m and and $\{(V_\alpha, y^l_\alpha)\}$ on N^n so that $f[U_\alpha] \subseteq V_\alpha$ for each α. If C is the set of critical points of f, then by the result for maps between Euclidean $f[C \cap U_\alpha]$ has measure zero in V_α and thus also in M. Therefore $f[C] = \bigcup_\alpha f[U_\alpha \cap C]$ is a countable union of sets of measure zero and thus is also a set of measure zero.

Now let $f : \mathbf{R}^m \to \mathbf{R}^n$ and assume that the result holds whenever the domain has dimension less than m. Write the map as $f = (f^1, \dots, f^n)$ where the functions f^l are the component functions of f in the standard coordinates on $\mathbf{R}^n$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$ (an m-tuple of non-negative integers) let $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_m^{\alpha_m}$ where $\partial_i = \partial/\partial x^i$ is the partial derisive by the ith coordinate function of $\mathbf{R}^m$.

Let $C \subset \mathbf{R}^m$ be the set of critical points of f. For each $\alpha \neq 0$ and each $l \in \{1, \dots, n\}$ the set

$$D_{\alpha,l} := \{x \in \mathbf{R}^m : \partial^\alpha f^l(x) = 0, \quad \text{there exists } i \quad \partial^i\partial^\alpha f^l(x) \neq 0\}$$

is a smooth hypersurface in $\mathbf{R}^n$. (This follows by applying the implicit function theorem to the function $\partial^\alpha f$.) By the induction hypothesis the set of critical values of $f|_{D_{\alpha,l}}$ has measure zero in $\mathbf{R}^n$. But the set of critical points of $f|_{D_{\alpha,l}}$ contains $C_{\alpha,l} := C \cap D_{\alpha,l}$. Therefore for each pair α, l the set $f[C_{\alpha,l}]$ has measure zero. If C_∞ is the subset of C of all x where $\partial^\alpha f^l(x) = 0$ for all $\alpha \neq 0$ and all $l \in \{1, \dots, n\}$, then $C = C_\infty \cup \bigcup_{\alpha,l} C_{\alpha,l}$. As this is a countable union and each $f[C_{\alpha,l}]$ has measure zero, it is enough to show that $f[C_\infty]$ has measure zero.

Toward this end let P be a closed cube in $\mathbf{R}^m$ with edges parallel to the axis and with sides of length one. We show $f[P \cap C_\infty]$ has measure zero. Let k be a positive integer so that $kn - m > 0$. The partial derivatives of each f^l of all orders vanish at points of C_∞ and the $k+1$-st partial derivatives are all continuous and thus bounded on the closed bounded set P. Therefore using the first $k+1$ terms of the power series expansion of the f^l's about $x \in P \cap C_\infty$ there is a constant c_0, only depending on f, k and P so that

$$\|f(x) - f(y)\| \leq c_0\|x - y\|^k \quad \text{for all } x \in P \cap C_\infty \text{ and all } y \in P$$

The cube P can be covered by l^m closed cubes with sides of length $1/l$ and sides parallel to the axis. Let $\mathcal{C}$ be the subset of these cubes that have at least one point in common with $C_\infty \cap P$. For any $P_i \in \mathcal{C}$ there is a point $x_i \in P_i \cap C_\infty$. As the cube P_i has diameter $\sqrt{m}/l$ the last inequality implies $f[P_i]$ is contained in a ball with radius $c_0(\sqrt{m}/l)^k$ centered at $f(x_i)$. Using the formula for the volume of a unit ball in $\mathbf{R}^n$ this shows there is a constant $c_1 = c_1(m, n, k, c_0)$ so that if $\mathcal{H}^n$ is the Lebesgue measure on $\mathbf{R}^n$

$$\mathcal{H}^n(f[P_i]) \leq c_1\left(\frac{1}{l^k}\right)^n = c_0\frac{c_1}{l^{kn}}.$$

As the set $\mathcal{C}$ contains at most l^m elements and it covers $P \cap C_\infty$ we have

$$\mathcal{H}^n(f[P \cap C_\infty]) \leq l^m c_1\frac{1}{l^{nk}} = \frac{c_1}{l^{kn-m}}.$$

But $kn - m > 0$ so letting $l \to \infty$ shows that $\mathcal{H}^n(f[P \cap C_\infty]) = 0$. But C_∞ can be covered by a countable collection of unit cubes, so $f[C_\infty]$ has measure zero. This completes the proof. $\square$

Exercise C.1.7. Use a packing argument like the one in the last part of the proof of Sard's theorem to show that if $f : \mathbf{R}^m \to \mathbf{R}^n$ is a smooth map and $m < n$, then $f[\mathbf{R}^m]$ has measure zero in $\mathbf{R}^n$. Then extend this to maps $f : M^m \to N^n$ between manifolds. (This completes the part of Sard's theorem omitted above.) *Hint*: Let $P \subset \mathbf{R}^m$ be a closed cube with sides parallel to the axis and of length one. As f is smooth and P compact there is a constant c_0 so that $\|f(x) - f(y)\| \leq c_0\|x - y\|$ for all $x, y \in P$. Then for each $l = 2, 3, \ldots$ the cube P can be covered by a collection $\mathcal{C}$ of l^m cubes with sides of length $1/l$. The image of $f[P_i]$ any cube $P_i \in \mathcal{C}$ will be a subset of a ball of radius $c_0\sqrt{m}/l$, and therefore $\mathcal{H}^n(f[P_i]) \leq c_1/l^n$. Thus thus $\mathcal{H}^n(f[P]) \leq l^m c_1/l^n \to 0$ as $l \to \infty$. Now cover $\mathbf{R}^m$ by a countable number of such cubes. □

C.2. Fiber Integrals and the Coarea Formula

Let M^m and N^n be smooth Riemannian manifolds. We will usually denote Riemannian metrics by $\langle\,,\rangle$ and trust to the context to make it clear which Riemannian metric is being referred to. If there is some chance of confusion the metrics on M^m and N^n will be written as $g^M(\,,)$ and $g^n(\,,)$. The length of a vector $X \in T(M)$ is denoted by $\|X\| := \sqrt{\langle X, X\rangle}$. If $X_1, \ldots, X_k \in T(M)_x$ then the length of the element $X_1 \wedge \cdots \wedge X_k \in \bigwedge^k T(M)_x$ (the k-th exterior power of $T(M)_x$) is

$$\|X_1 \wedge \cdots \wedge X_k\|^2 = \det(\langle X_i, X_j\rangle).$$

The geometric interpretation of this is that $\|X_1 \wedge \cdots \wedge X_k\|$ is the volume of the parallelepiped spanned by $X_1, \ldots, X_k$ (that is the set of vectors of the form $t_1X_1 + \cdots + t_kX_k$ where $0 \leq t_i \leq 1$.)

We now define the ***Jacobian*** of $f : M^m \to N^n$ separately in the cases $m \leq n$ and $m \geq n$ (the definitions agree when$m - n$). While we are most interested in the case where $m \geq n$ we first discuss that case where $m \leq n$ as it is more familiar. In the case $m \leq n$ the Jacobian of f at x is defined by

$$Jf(x) := \|f_*e_1 \wedge \cdots \wedge f_*e_m\|$$

where $e_1, \ldots, e_m$ is an orthonormal basis of $T(M)_x$. This is easily seen to be independent of the choice of the orthonormal basis $e_1, \ldots, e_m$. The geometric meaning of $Jf(x)$ is the dilation factor of the area element of N under f_*. While this definition is easy to use in proving things about the Jacobian and makes the its geometric meaning clear it is not the easiest to use in calculation as one has to find an orthonormal basis of $T(M)_x$. There is anther formula for $J(f)$ which gets around this. Let f^t_{*x} be the transpose of f_{*x}. That is $f^t_{*x} : T(N)_{f(x)} \to T(M)_x$ defined by $g^N(X, f^t_{*x}Y) = g^M(f_{*x}X, Y)$ for $X \in T(M)_x$ and $T(N)_{f(x)}$. We then leave it as an exercise to show $Jf(x)$ is also given by

$$Jf(x) = \sqrt{\det(f^t_{*x}f_{*x})}.$$

This is used, just as various special case of it are in calculus, to compute the area of submanifolds. Let $f : M^m \to N^n$ be an injective map between Riemannian manifolds manifolds (which implies $m \leq n$). Then the m-dimensional surface area of $f[M]$ is given by

$$\mathcal{H}^m(f[M]) = \int_M Jf(x)\,dx$$

where dx is the Riemannian volume measure on M^m. When $m = 1$ this reduces to the usual formula $\mathrm{Length}(f) = \int_a^b \partial'(t)\|\,dt$ for the length of a curve $f : [a, b] \to N^n$ and if $U \subset \mathbf{R}^2$ is an open set and $f : U \to \mathbf{R}^3$ then the Jacobian is given by $Jf = \|\partial f/\partial u \times \partial f/\partial v\|$ so the last displayed formula reduces to the usual formula for computing the area of surfaces in space.

We now define the Jacobian of $f : M^m \to N^n$ in the case $m \geq n$. In this case the

$$Jf(x) = \begin{cases} 0, & \text{if } x \text{ is a critical point of } f, \\ \|f_*e_i \wedge \cdots \wedge f_*e_n\|, & \begin{cases} \text{if } x \text{ is a regular value of } f \text{ and} \\ e_1, \ldots, e_n \text{ is an orthonormal} \\ \text{of } \mathrm{Kernel}(f_{*x})^{\perp}. \end{cases} \end{cases}$$

Note that $Jf(x) \neq 0$ if and only if x is a regular value of f. As before there it is possible to express this in terms of the transpose of f_{*x},

$$Jf(x) = \sqrt{\det(f_{*x}f_{*x}^t)}.$$

(Here the factors are in the opposite order than the case where $m \leq n$.) In the case that N^n is oriented and there is anther useful formula for $Jf(x)$. Let Ω_N be the volume form of N and x a regular point of f. Let $e_1, \ldots, e_{m-n}$ be an orthonormal basis of $\mathrm{Kernel}(f_{*x})^{\perp}$. Then a chase through the definitions shows that

$$Jf(x) = |f^*\Omega_{f(x)}(e_1, \ldots, e_{m-n})| = |\Omega_N(f_*e_1, \ldots, f_*e_{m-n})|$$

The basic result on integration over fibers of smooth maps between Riemannian manifolds is

Theorem C.2.1 (The Coarea Formula, Federer [**10**, 1959]). *Let $f : M^m \to M^n$ be a smooth map between Riemannian manifolds with $m \geq n$. Then for almost every $y \in N^n$ the fiber $f^{-1}[y]$ either empty or a smooth imbedded submanifold of M^m of dimension $m - n$. For each regular value y of f let dA be the $m - n$-dimensional surface area measure on $f^{-1}[y]$. Then for any Borel measurable function h on M^m*

$$\int_{N^n} \int_{f^{-1}[y]} h\,dA\,dy = \int_{M^m} h(x)Jf(x)\,dx$$

where dy is the Riemannian volume measure on N^n and dx is the Riemannian volume measure on M^m. If $\mathcal{H}^{m-n}(f^{-1}[y])$ is the $m - n$-dimensional surface area measure of $f^{-1}[y]$ then letting $h \equiv 1$ implies

$$\int_{N^n} \mathcal{H}^{m-n}(f^{-1}[y])\,dy = \int_{M^m} Jf(x)\,dx$$

Before giving the proof we state some special cases where the result should look either familiar or at least more concrete. First if $M^m = P^{m-n} \times N^n$ is a product manifold and $f(x,y) = y$ is the projection onto the second factor, then Jacobian is easily seen to be $Jf \equiv 1$ and in this case the coarea formula just reduces to Fubini's theorem on repeated integrals. Thus one way to view the coarea formula is as a generalization of Fubini's theorem to the curved setting.

As anther example note if $f : M^m \to \mathbf{R}$ and ∇f is the gradient of f. (That is the vector field so that for and vector tangent to M^m there holds $\langle \nabla f, X\rangle = df(X)$). Then ∇f is perpendicular to the fibers (level sets) of f and thus at regular points $\nabla f / |\|\nabla f\|$ is an orthonormal basis of $T(f^{-1}[y])^\perp$ (where $y = f(x)$). Thus $Jf(x) = df(\nabla f/\|\nabla f\|) = \|\nabla f\|$. So in this case the coarea formula with $h \equiv 1$ becomes

$$\int_{-\infty}^{\infty} \text{Area}\{x : f(x) = t\}\, dt = \int_M \|\nabla f\|\, dA.$$

This formula is useful proving inequalities of Sobolev type.

Remark C.2.2. Federer proves the coarea formula in a much more general setting where $f : M^m \to N^n$ is Lipschitz. The simpler proof for smooth functions is taken from [**19**, Appendix pp. 66–68]. □

C.3. The Lemma on Fiber Integration

Let $f : M^m \to N^n$ be a smooth map between manifolds with M^m and N^n oriented. If $m \geq n$ then near any regular point x of f the fiber $f^{-1}[y]$ (with $y = f(x)$) will be given the orientation so that local near x the manifold M^m looks like

$$M^m \approx \text{fiber} \times \text{base}.$$

To be more precise let $v_1, \dots, v_n$ be the an oriented basis of $T(N)_y$ and let $V_1, \dots, V_n \in T(M)_x$ be any vectors so that $f_* V_i = v_i$. Then a basis $X_1, \dots, X_{m-n}$ of $T(f^{-1})[y])_x = \text{Kernel}(f_{*x})^\perp$ is oriented if and only if the basis $X_1, \dots, X_{m-n}, V_1, \dots, V_n$ agrees with the orientation of M^m.

Lemma C.3.1 (Lemma on fiber integration). *Let $f : M^m \to N^n$ be a smooth map between oriented manifolds with $m \geq n$. Let α be a smooth compactly supported $(m-n)$ form on M^m and β a smooth n form on N^n. With the above convention on the orientation of fibers*

$$\int_{N^n} \left(\int_{f^{-1}[y]} \alpha \right) \beta(y) = \int_{M^m} \alpha \wedge f^* \beta.$$

Proof. We first consider the case $M^m = \mathbf{R}^m$, $N^n = \mathbf{R}^n$ and

$$f(x^1, \dots, x^m) = (x^{m-n+1}, \dots, x^m).$$

That is f is projection onto the last n coordinates. In these coordinates

$$\alpha = \sum_{i_1 < \cdots < i_{n-m}} a_{i_1 \cdots i_{m-n}}(x^1, \dots, x^m) dx^{i_1} \wedge \cdots \wedge dx^{i_{m-n}}$$

where each $a_{i_1 \cdots i_{m-n}}$ is smooth and compactly supported. Likewise β is given by

$$\beta = b(y^1, \dots, y^n) dy^1 \wedge \cdots \wedge dy^n$$

Then for $y = (y^1, \dots, y^n) \in \mathbf{R}^n$ the restriction of α to the tangent bundle of $f^{-1}[y]$ is

$$\alpha|_{f^{-1}[y]} = a_{1 \cdots (m-n)}(x^1, \dots, x^{m-n}, y^1, \dots, y^n) dx^1 \wedge \cdots \wedge dx^{m-n}$$

Thus

$$\int_{f^{-1}[y]} \alpha = \int_{\mathbf{R}^{m-n}} a_{1 \cdots (m-n)}(x^1, \dots, x^{m-n} y^1, \dots, y^n) dx^1 \cdots dx^{m-n},$$

which implies

$$\begin{aligned} &\int_{\mathbf{R}^n} \left(\int_{f^{-1}[y]} \alpha \right) \beta(y) \\ &\quad = \int_{\mathbf{R}^n} \int_{\mathbf{R}^{m-n}} a_{1 \cdots (m-n)}(x^1, \dots, x^{m-n}, y^1, \dots, y^n) dx^1 \cdots dx^{m-n} \\ &\qquad\qquad \times b(y^1, \dots, y^n) dy^1 \cdots dy^n \\ &\quad = \int_{\mathbf{R}^m} a_{1 \cdots (m-n)}(x^1, \dots, x^m) b(x^{m-n+1}, \dots, x^m) dx^1 \cdots dx^m \qquad \text{(C.1)} \end{aligned}$$

On the other hand

$$f^*\beta = b(x^{m-n-1}, \dots, x^m)\, dx^{m-n+1} \wedge \cdots \wedge dx^m$$

and

$$\alpha \wedge f^*\beta = a_{1 \cdots (m-n)}(x^1, \dots, x^m) b(x^{m-n+1}, \dots, x^m)\, dx^1 \wedge \cdots \wedge dx^m$$

$$\int_{\mathbf{R}^m} \alpha \wedge f^*\beta = \int_{\mathbf{R}^m} dx^1 \cdots dx^m.$$

Comparing this with equation (C.1) shows the lemma holds in the special case.

In the general case of a smooth map $f : M^m \to N^n$ between oriented manifolds, let M^* be the set of regular points of f. Then M^* is an open subset of M^m and if $x \notin M^*$ then x is a critical value of f and $(f^*\beta)_x = 0$. Thus $\alpha \wedge f^*\beta = 0$ on $M \setminus M^*$ and

$$\int_M \alpha \wedge f^*\beta = \int_{M^*} \alpha \wedge f^*\beta.$$

By Sard's theorem $f^{-1}[y] \subset M^*$ for almost all $y \in N^n$ so

$$\int_{N^n} \left(\int_{M^* \cap f^{-1}[y]} \alpha \right) \beta(y) = \int_{N^n} \left(\int_{f^{-1}[y]} \alpha \right) \beta(y).$$

At every point $x_0 \in M^*$ we can use the implicit function theorem to find coordinates $(x^1, \dots, x^m)$ centered at x_0 and $(y^1, \dots, y^n)$ centered at $y_0 = f(x_0)$ so that these coordinates agree with the orientations of M^m and N^n and in these coordinates f is given by

$$f((x^1, \dots, x^m) = (x^{m-n+1}, \dots, x^m)$$

(which to be absolutely correct should be written as $y^k(f(p)) = x^{m-n+k}(p)$). Thus if the support of α is in the domain of the chart $(x^1, \dots, x^m)$, then then lemma holds as it reduces to the special case we have already considered. The general case now follows by a partition of unit argument and the observation that we only need to integrate over the set of regular points. □

Proof of the Coarea Formula. We first consider the case where both of M^m and N^n are oriented, the map $f : M^m \to N^n$ is a submersion, and the function h is smooth and compactly supported. Let Ω_M and Ω_N be the volume forms of M and N. As f is a submersion every point $x \in M^m$ is a regular point of f. Thus it is possible to choose and orthonormal basis $e_1, \dots, e_m$ of $T(M)_x$ so that $e_1, \dots, e_{m-n}$ is a basis of $\operatorname{Kernel}(f_{*x}) = T(f^{-1}[y])_x$ (where $y = f(x)$) and the vectors $e_{m-n+1}, \dots, e_m$ are an orthogonal basis of $\operatorname{Kernel}(f_{*x})^\perp$ such that the orientation of the basis $f_*e_{m-n+1}, \dots, f_*e_m$ of $T(N)_{f(x)}$ agrees with the orientation of N. Let $\sigma^1, \dots, \sigma^m$ be the one forms dual to $e_1, \dots, e_m$. Define an $(m-n)$-form ω_1 and an n-form ω_2 by

$$\omega_1 := \sigma^1 \wedge \cdots \sigma^{m-n} \qquad \omega_2 := \sigma^{m-n+1} \wedge \cdots \wedge \sigma^m.$$

These forms are defined independent of the choice of the choice of the orthonormal basis $e_1, \dots, e_m$ and thus they are smooth forms on all of M^m. Also, form our convention on the orientation of fibers, the restriction of ω_1 to a fiber is the volume form on the fiber.

Either from a direct calculation or an earlier formula

$$f^*\Omega_N(e_{m-n+1}, \dots, e_m) = \Omega_N(f_*e_{m-n+1}, \dots, f_*e_m) = Jf(x)$$

and $f_*e_i = 0$ if $i \le m - n$. Thus

$$f^*\Omega_N = (Jf)\omega_2.$$

But $\omega_1 \wedge \omega_2 = \sigma^1 \wedge \cdots \wedge \sigma^m = \Omega_M$ is the volume form on 0n M. Therefore the last formula implies

$$\omega_1 \wedge f^*\omega_2 = (Jf)\Omega_M = J(f)\, dV,$$

which is an infinitesimal version of the coarea formula. Now apply the lemma on fiber integration to the forms $\alpha = h\omega_1$ and $\beta = \Omega_N$ and rewrite the integral $\int_N \int_{f^{-1}[y]} h\, dA\, dy$ and $\int_M h(x)Jf(x)\, dV$ in terms of these forms to get

$$\begin{aligned}\int_N \int_{f^{-1}[y]} h\, dA\, dy &= \int_N \int_{f^{-1}[y]} h\omega_1\, \Omega_N(y)\\ &= \int_M h\omega_1 \wedge f^*\Omega_N\\ &= \int_M h(x)Jf(x)\, dx\end{aligned}$$

which is exactly the coarea formula in this case.

In the case of a general smooth map $f : M^m \to N^n$ between smooth Riemannian manifolds let, as in the proof of the lemma on fiber integration, $M^* = \{x : Jf(x) \neq 0\}$ be the set of regular points of f. Then

$$\int_M h(x)Jf(x)\, dx = \int_{M^*} h(x)Jf(x)\, dx.$$

Also as $f^{-1}[y] \subset M^*$ for almost all $y \in N^n$

$$\int_N \int_{M^* \cap f^{-1}[y]} h\, dA\, dy = \int_N \int_{f^{-1}[y]} h\, dA\, dy$$

Thus we can replace M by M^* and assume that f is a submersion. If U is an open orientable open subset of M^m so that $f[U]$ for some orientable open subset V of N^n and h is smooth and compactly supported inside of U, then the coarea formula follows form the case we have already done. As M can be covered by such open sets U, a partition of unity argument shows that the coarea formula holds for general smooth h. The extension to Borel measurable functions now follows by standard approximation arguments. $\square$

C.4. Remarks on the coarea formula and fiber integration

The restriction to smooth function in the coarea formula is not necessary and in many contexts not natural. For a general version that covers most case that we would need is for Lipschitz maps. Recall that if $f : M^m \to N^n$ is a Lipschitz map between Riemannian manifolds then by a theorem due to Rademacher the derivative f_* of f exists almost everywhere on M^m. (This follows from the usual version of Rademacher's theorem for Lipschitz maps between Euclidean spaces.) Thus the Jacobian $J(f)$ is defined almost everywhere on M^m. The following general result is due to Federer (who was the first to state the coarea formula).

THEOREM C.4.1 (Lipschitz coarea formula [**10**, 1959]). *Let $f : M^m \to N^n$ be a Lipschitz map between Riemannian manifolds with $m \geq n$. Then for almost all $y \in N^n$ the fiber $f^{-1}[y]$ has local finite $(m-n)$-dimensional Hausdorff measure and for any Borel measure function h on M^m*

$$\int_{N^n} \int_{f^{-1}[y]} h\, d\mathcal{H}^{m-n}(y) = \int_{M^m} h(x)Jf(x)\, dx$$

where $\mathcal{H}^{m-n}$ is the $(m-n)$-dimensional Hausdorff on M^m.

There will be times in the next section where the coarea formula is applied to functions that are not smooth. In these cases the last result will always apply. We can use approximation arguments to prove results about non-smooth functions by applying the coarea formula to smooth functions and taking limits when we are done. As this gets tedious and there are no real ideas involved, we will just apply the coarea formula directly to what ever seems appropriate. As an example of this let f be a smooth compactly supported function on $\mathbf{R}^n$ smooth function. Then we will want to use the coarea formula in the form

$$\int_0^\infty A(\partial\{x : |f(x)| \geq t\})\,dt = \int_{\mathbf{R}^n} \|\nabla|f|(x)\|\,dx = \int_{\mathbf{R}^n} \|\nabla f(x)\|\,dx$$

where dA is the surface area measure on the boundary of $\{x : |f(x)| \geq t\}$. As $|f|$ is not smooth our form of the coarea does not apply directly. But $|f|$ is a Lipschitz function it is covered by Federer's result. Similar remarks hold for the lemma of fiber integration which can be proven in much more generality.

APPENDIX D

Isoperimetric Constants and Sobolev Inequalities

D.1. Relating Integrals to Volume and Surface Area

Let M^m be Riemannian manifold. If f is a smooth function on M^m denote by ∇f the gradient vector field of f. In this section our goal is to understand when inequalities of the type

$$\left(\int_M |f|^q\, dV\right)^{\frac{1}{q}} \le C \int_M \|\nabla f\|\, dV \tag{D.1}$$

or more generally of the form

$$\left(\int_M |f|^q\, dV\right)^{\frac{1}{q}} \le C \left(\int_M \|\nabla f\|^p\, dV\right)^{\frac{1}{p}}$$

hold for all f in the space $C_0^\infty(M)$ of infinitely differentiable functions with compact support.

The basic idea behind the proofs are as follows. Let V be the Riemannian volume measure on M and A is the surface area measure on hypersurfaces. Then for any measurable function h on M there is the basic identity

$$\int_M |h|\, dV = \int_0^\infty V\{x : |h(x)| \ge s\}\, ds. \tag{D.2}$$

and there is also the coarea formula which we write in the form

$$\int_M \|\nabla f\|\, dA = \int_0^\infty A(\partial\{x : |f(x)| \ge t\})\, dt.$$

Using these formulas it is possible to relate isoperimetric type inequalities $V(D) \le cA(\partial D)^\alpha$ directly to integral inequalities of the type (D.1). It even turns out that the best constant in the isoperimetric inequality gives the best constant in the corresponding analytic inequality.

Exercise D.1.1. Prove the formula (D.2). □

Let M^m be a non-compact Riemannian manifold. As one of the cases of interest is when M^m is a domain in $\mathbf{R}^m$ we do not assume that M^m is complete. Say that M^m satisfies as ***isoperimetric inequality of degree*** α if and only if there is a constant c so that for all domains $D \subset\subset M$ with smooth boundary

$$V(D) \le cA(\partial D)^\alpha.$$

($D \subset\subset M$ means that the closure of D is a compact subset of the interior of M.) If such an inequality holds then the smallest such constant $h_\alpha = h_\alpha(M)$ is the ***isoperimetric constant of degree*** α for M^m. That is

$$h_\alpha(M) = \sup_{D \subset\subset M} \frac{V(D)}{A(\partial D)^\alpha}$$

when this is finite. Fix $x \in M$ and let $B(x,r)$ be the geodesic ball of radius r about x. For small r we have $V(B(x,r)) = c_1 r^m + O(r^{m+1})$ and $A(\partial B(x,r)) = c_2 r^{m-1} + O(r^m)$. Using this in the definition of $h_\alpha(M)$ yields

$$h_\alpha(M) < \infty \quad \text{implies} \quad \alpha \le \frac{m}{m-1} \quad (m = \dim(M)).$$

The most obvious example of a manifold that satisfies an isoperimetric inequality is $\mathbf{R}^m$ where the usual isoperimetric inequality implies

$$h_{\frac{m}{m-1}}(\mathbf{R}^n) = \frac{V(B^m)}{A(S^{m-1})^{\frac{m}{m-1}}}.$$

Exercise D.1.2. Use the last equation to show that if M is a domain in $\mathbf{R}^m$ of finite volume V_0 (but not necessary bounded) then for all $0 < \alpha \le m/(m-1)$

$$h_\alpha(M) \le \left(h_{\frac{m}{m-1}}(\mathbf{R}^n)\right)^{\frac{(m-1)\alpha}{m}} V_0^{1-\frac{(m-1)\alpha}{m}}.$$

Thus M satisfies isoperimetric inequalities of all degrees α with $0 < \alpha \le m/(m-1)$. □

Exercise D.1.3. If M^m satisfies isoperimetric inequalities of degree α and of degree β then it also satisfies isoperimetric inequalities of degree γ for all $\alpha \le \gamma \le \beta$ and

$$h_\gamma(M) \le h_\alpha(M)^{\frac{\gamma-\beta}{\alpha-\beta}} h_\beta(M)^{\frac{\alpha-\gamma}{\alpha-\beta}}.$$

□

D.2. Sobolev Inequalities

Theorem D.2.1 (Federer-Fleming [**11**, 1960], Yau [**31**, 1975]). *Assume M^m satisfies an isoperimetric inequality of degree α with $1 \le \alpha \le m/(m-1)$. Then for $f \in C_0^\infty(M)$ the Sobolev inequality*

$$\int_M |f|^\alpha \, dV \le h_\alpha(M) \left(\int_M \|\nabla f\| \, dV \right)^\alpha$$

holds. More over this is sharp in the strong sense that if an inequality $\int |f|^\alpha \, dV \le c \left(\int \|\nabla f\| \, dV\right)^\alpha$ holds for all $f \in C_0^\infty(M)$ then M satisfies an isoperimetric inequality of degree α and $h_\alpha(M) \le c$.

Remark D.2.2. Federer-Fleming [**11**] gave this result (and proof) in the case M^m is Euclidean space. Yau [**31**] showed that the same proof extends to Riemannian manifolds.

PROOF. Let $f \in C_0^\infty(M)$ and set $V(t) = V\{x : |f(x)| \geq t\}$. Then equation (D.2) and a change of variable

$$\begin{aligned}
\int |f|^\alpha \, dV &= \int_0^\infty V\{x : |f(x)|^\alpha \geq s\} \, ds \\
&= \alpha \int_0^\infty V\{x : |f(x)|^\alpha \alpha t^\alpha\} t^{\alpha-1} \, dt \\
&= \alpha \int_0^\infty V(t) t^{\alpha-1} \, dt.
\end{aligned}$$

By the coarea formula and that M satisfies an isoperimetric inequality of degree α

$$\begin{aligned}
\int \|\nabla f\| dV &= \int_0^\infty A(\partial\{x : |f(t)| \geq t\}) \, dt \\
&\geq \frac{1}{h_\alpha(M)^{\frac{1}{\alpha}}} \int_0^\infty V(t)^{\frac{1}{\alpha}} \, dt
\end{aligned}$$

So it is enough to show

$$\alpha \int_0^\infty V(t) t^{\alpha-1} \, dt \leq \left(\int_0^\infty V(t)^{\frac{1}{\alpha}}, dt \right) \tag{D.3}$$

Let

$$F(s) = \alpha \int_0^s V(t) t^{\alpha-1} \, dt$$

so that

$$F'(s) = \alpha V(s) s^{\alpha-1}.$$

Also let

$$G(s) = \left(\int_0^s V(t)^{\frac{1}{\alpha}} \, dt \right).$$

Using that $V(t)$ is monotone decreasing (so that $\int_0^s V(t)^{\frac{1}{\alpha}} \, dt \geq sV(s)^{\frac{1}{\alpha}}$) and $\alpha \geq 1$

$$\begin{aligned}
G'(s) &= \alpha \left(\int_0^s V(t)^{\frac{1}{\alpha}} \, dt \right)^{\alpha-1} V(s)^{\frac{1}{\alpha}} \\
&\geq \alpha \left(sV(s)^{\frac{1}{\alpha}} \right)^{\alpha-1} V(s)^{\frac{1}{\alpha}} \\
&= F'(s)
\end{aligned}$$

As $F(0) = G(0) = 0$ this implies $F(s) \leq G(s)$ for all $s \geq 0$. Letting $s \to \infty$ then shows that (D.3) holds and completes the proof of the Sobolev inequality.

To see that $h_\alpha(M)$ is the sharp constant assume an inequality

$$\int |f|^\alpha \, dV \leq c \left(\int \|\nabla f\| \, dV \right)^\alpha \tag{D.4}$$

holds for all $f \in C_0^\infty(M)$. Then by approximation this inequality also holds for all compactly supported Lipschitz functions. Let $D \subset\subset M$ have smooth boundary and let $\rho_D(x) = \operatorname{dist}(x, D)$. Define a function a Lipschitz f_ε by

$$f_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in D, \\ 1 - \rho_D(x)/\varepsilon, & \text{if } 0 < \rho_D(x) < \varepsilon, \\ 0, & \text{if } \varepsilon \le \rho_D(x) \end{cases}$$

Let $\tau_\varepsilon(D) := \{x \in M^m : 0 < \rho_D(x) < \varepsilon\}$. Then

$$\|\nabla f_\varepsilon(x)\| = \begin{cases} \frac{1}{\varepsilon}, & \text{if } x \in \tau_\varepsilon(D) \\ 0, & \text{otherwise.} \end{cases}$$

Also $V(\tau_\varepsilon(D)) + \varepsilon A(\partial D) + O(\varepsilon^2)$. Thus letting $\varepsilon \searrow 0$

$$\int \|\nabla f_\varepsilon\| \, dV = \frac{1}{\varepsilon} V(\tau_\varepsilon(D)) \to A(\partial D), \qquad \int |f_\varepsilon|^\alpha \, dV \to V(D).$$

Using these relations in (D.4) implies $V(D) \le cA(\partial D)^\alpha$. Thus $h_\alpha(M) \le c < \infty$. This completes the proof. $\square$

D.3. McKean's and Cheeger's lower bounds on the first eigenvalue

THEOREM D.3.1 (Cheeger [**7**, 1970]). *If $h_1(M) < \infty$ then for each $1 < p < \infty$ and every $f \in C_0^\infty(M)$*

$$\left(\int |f|^p \, dV \right)^{\frac{1}{p}} \le p h_1(M) \left(\int \|\nabla f\|^p \, dV \right)^{\frac{1}{p}} \tag{D.5}$$

In particular when $p = 2$ this implies

$$\int \|\nabla f\|^2 \, dV \ge \frac{1}{4h_1(M)^2} \int f^2 \, dV$$

Thus $1/(4h_1(M)^2)$ is a lower bound for the first Eigenvalue for the Laplacian on M.

REMARK D.3.2. The number $1/h_1(M)$ is often called the ***Cheeger constant*** of the manifold.

PROOF. We first consider the case $p = 1$. Set $u \in C_0^\infty(M)$ and set $V(t) = V\{x : |u(x)| \ge t\}$ and $A(t) = A(\partial\{x : |u(x)| \ge t\})$. By the definition of $h_1(M)$ the inequality $V(t) \le h_1(M)A(t)$ holds. Using the equality (D.2) and the coarea formula

$$\int_M |u| \, dV = \int_0^\infty V(t) \, dt \le h_1(M) \int_0^\infty A(t) \, dt = \int_M \|\nabla u\| \, dV.$$

If $1 < p < \infty$ let $u = |f|^p$. Then $\|\nabla u\| = p|f|^{p-1}\|\nabla f\|$. Use this u in the last inequality and Hölder's inequality with the conjugate exponents p and

$p' = p/(p-1)$.

$$\int_M |f|^p \, dV \le p h_1(M) \int_M |f|^{p-1} \|\nabla f\| \, dV$$
$$\le p h_1(M) \left(\int_M |f|^p \, dV \right)^{\frac{1}{p'}} \left(\int_M \|\nabla f\|^p \, dV \right)^{\frac{1}{p}}.$$

Dividing by $(\int |f|^p \, dV)^{1/p'}$ completes the proof. □

PROPOSITION D.3.3. *If M^m is a complete simply connected Riemannian manifold with all sectional curvatures $\le -K_0 < 0$ the isoperimetric constant $h_1(M)$ satisfies*

$$h_1(M^m) \le \frac{1}{(m-1)\sqrt{K_0}}$$

This estimate is sharp on the hyperbolic space of constant sectional curvature $-K_0$.

PROOF. Let $D \subset\subset M$ have smooth boundary and let $x_0 \in M^m$ with $x_0 \notin D$. Then the function ρ smooth on D. As ρ is the distance from a point $\|\nabla \rho\| \equiv 1$. From the Bishop comparison theorem the Laplacian $\Delta\rho$ of ρ satisfies

$$\Delta \rho \ge (m-1)\sqrt{K_0}.$$

Let η be the out ward unit normal to ∂D. Then by the last inequality and the divergence theorem

$$(m-1)\sqrt{K_0} V(D) \le \int_D \Delta\rho \, dV = \int_{\partial D} \langle \nabla \rho, \eta \rangle \, dA \le A(\partial D).$$

Thus $h_1(M) \le 1/((m-1)\sqrt{K_0})$ as claimed.

To verify the claim about hyperbolic H^m space we normalize so that $K_0 = 1$. If $B(r)$ is a geodesic ball in the hyperbolic space of dimension m, then

$$V(B(r)) = A(S^{m-1}) \int_0^r \sinh^{m-1}(t) \, dt = A(S^{m-1}) \frac{e^{(m-1)r}}{2^m(m-1)} + O(e^{(m-2)r})$$

$$A(\partial B(r)) = A(S^{m-1}) \sinh^{m-1}(r) = A(S^{m-1}) \frac{e^{(m-1)r}}{2^m} + O(e^{(m-2)r}).$$

Therefore $\lim_{r\to\infty} V(B(r))/A(\partial B(r)) = 1/(m-1)$ so that $h_1(H^m) \ge 1/(m-1)$. As we already have the inequality $h_1(H^m) \le 1/(m-1)$, this completes the proof. □

THEOREM D.3.4 (McKean [**22**, 1970]). *Let M^m be a complete simply connected manifold with sectional curvatures $\le -K_0 < 0$. Then for any $f \in C_0^\infty(M)$ and $1 \le p < \infty$*

$$\int_M |f|^p \, dV \le \frac{1}{p(m-1)\sqrt{K_0}} \int_M \|\nabla f\|^p \, dV$$

and thus the first eigenvalue of any $D \subset\subset M^m$ satisfies

$$\lambda_1(D) \geq \frac{(m-1)^2 K_0}{4}$$

PROOF. This follows at once from the previous results. □

REMARK D.3.5. It is worth noting that the basic Sobolev inequality

$$\left(\int |u|^\alpha \, dV \right)^{\frac{1}{\alpha}} \leq C \int \|\nabla u\| \, dV$$

implies a large number of other inequalities just by use of the Hölder inequality and some standard tricks. For example if in the last inequality we replace u by $|f|^\beta$ where $\beta \geq 1$ is to be chosen latter,

$$\begin{aligned} \left(\int |f|^{\alpha\beta} \, dV \right)^{\frac{1}{\alpha}} &\leq C\beta \left(\int |f|^{\beta-1} \|\nabla f\| \, dV \right) \\ &\leq C\beta \left(\int |f|^{(\beta-1)p'} \, dV \right)^{\frac{1}{p'}} \left(\int \|\nabla f\|^p \, dV \right)^{\frac{1}{p}}. \end{aligned}$$

now choose β so that $(\beta - 1)p' = \alpha\beta$, that is $\beta = p/(p - \alpha(p-1))$. Then the last inequality reduces to

$$\left(\int |f|^{\frac{\alpha p}{p-\alpha(p-1)}} \, dV \right)^{\frac{p-\alpha(p-1)}{\alpha p}} \leq \frac{Cp}{p - \alpha(p-1)} \left(\int \|\nabla f\|^p \, dV \right)^{\frac{1}{p}}.$$

For this to work we need $\beta \geq 1$ which implies $p < \alpha/(\alpha - 1)$. When $\alpha = m/(m-1)$, as it is in Euclidean space $\mathbf{R}^m$, the restriction on p is then $p < m$. It is not hard to check that all dilation invariant inequalities of the form $(\int |f|^q)^{\frac{1}{q}} \leq \text{Const.}(\int \|\nabla f\|^p)^{\frac{1}{p}}$ can be derived form the basic Sobolev inequality $\int |f|^{\frac{m}{m-1}} \, dV \leq h_{m/(m-1)} \left(\int \|\nabla f\| \, dV \right)^{\frac{m}{m-1}}$ in this manner. However, due to the application of the Hölder inequality, the constants in the resulting inequalities are no longer sharp.

D.4. Hölder Continuity

In applications a very important fact about the various Sobolev space $W^{1,p}(M^m)$ (this space is the completion of the space of smooth functions with the norm $\|f\|_{W^{1,p}} = (\int |f|^p \, dV)^{\frac{1}{p}} + (\int \|\nabla\|^{\frac{1}{p}} \, dV)^{\frac{1}{p}}$) are continuous when $p > m$. For functions defined on the line $\mathbf{R}$ this is easy to see from Hölder's inequality:

$$|f(x) - f(y)| \leq \int_x^y |f'(t)| \, dt \leq |x-y|^{\frac{1}{p'}} \left(\int_x^y |f'(t)|^p \, dt \right)^{\frac{1}{p}}.$$

We will now show that by an appropriate integral geometric trick this proof can be extended to higher dimensions. The idea is to connect two points in the domain of the function in question by an $(m-1)$-dimensional family of curves, do exactly the above calculation on each of the curves, and then integrate over the space of parameters. The coarea formula (in this case really only the change of variable formula in an integral) is used in computing the integrals.

THEOREM D.4.1. *Let $M^m \subseteq \mathbf{R}^m$ be an open set and $P, Q \in M$. Let $C = \frac{1}{2}(P+Q)$ be the center of the segment between P and Q let $r = \frac{1}{2}\|P-Q\|$ and $B(C,r)$ the ball with center C and radius r (this is the smallest ball containing both P and Q. Assume $B(C,r) \subseteq M$ and that $p > m$. Then for every $f \in C^\infty(M)$*

$$|f(P) - f(Q)| \le c\|P-Q\|^{1-\frac{m}{p}} \left(\int_{B(C,r)} \|\nabla f\|^p \, dV \right)^{\frac{1}{p}} \tag{D.6}$$

where

$$c \le \sqrt{2} A(B^{m-1})^{\frac{p-1}{p}} B\left(\frac{p-m}{p-1}, \frac{p-m}{p-1} \right)^{\frac{p}{p-1}}. \tag{D.7}$$

Here $A(B^{m-1})$ is the $(m-1)$-dimensional volume of the unit ball in $\mathbf{R}^{m-1}$ and $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt$ is the Beta function.

PROOF. If $P = Q$ then there is nothing to prove so assume that $P \ne Q$. Let B^{m-1} be the unit ball in the hyperplane $(P-Q)^\perp$ orthogonal to the vector $P - Q$. Define a map $\varphi : [0,1] \times B^{m-1} \to \mathbf{R}^m$ by

$$\varphi(t,x) = tP + (1-t)Q + t(1-t)\|P-Q\|x.$$

It is easily checked that the image $\operatorname{Im}\varphi$ of φ is contained in $B(C,r)$. Think of $[0,1] \times B^{m-1}$ as a subset of $\mathbf{R}^m = \mathbf{R} \times (P-Q)^\perp$. If $e_1, \ldots, e_{m-1}$ is an orthonormal basis of $(P-Q)^\perp$ then $\partial/\partial t, e_1, \ldots, e_{m-1}$ is an orthogonal basis of $T([0,1] \times B^{m-1})_{(t,x)}$ and

$$\varphi_* \frac{\partial}{\partial t} = \frac{\partial \varphi}{\partial t} = P - Q + (1-2t)\|P-Q\|x,$$

$$\varphi_* e_i = \|P-Q\| e_i.$$

As x is in the span of $e_1, \ldots, e_{m-1}$ this implies the Jacobian of φ is

$$J(\varphi) = \|P-Q\|^m (t(1-t))^{m-1}.$$

Thus for any function h defined on the image of φ the change of variable formula implies

$$\int_{\operatorname{Im}\varphi} h \, dV = \int_{B^{m-1}} \int_0^1 h(\varphi) J(\varphi) dt \, dx$$

We also note as $\|x\| \le 1$ that

$$\left\| \frac{\partial \varphi}{\partial t} \right\| \le \sqrt{2}\|P-Q\|.$$

For each $x \in B^{m-1}$ the curve $t \mapsto \varphi(t,x)$ starts at Q and ends at P and therefore by the fundamental theorem of calculus $\int_0^1 \partial/\partial t f(\varphi(t,x))\, dt =$

$f(Q) - f(P)$. Using the formulas and inequalities above and the Hölder inequality with exponents p and $p' = p/(p-1)$:

$$
\begin{aligned}
A(B^{m-1})|f(P) - f(Q)| &= \int_{B^{m-1}} |f(P) - f(Q)|\, dx \\
&\leq \int_{B^{m-1}} \int_0^1 \left| \frac{\partial}{\partial t} f(\varphi) \right| dt\, dx \\
&= \int_{B^{m-1}} \int_0^1 \left| \nabla f(\varphi) \cdot \frac{\partial \varphi}{\partial t} \right| dt\, dx \\
&\leq \sqrt{2}\|P - Q\| \int_{B^{m-1}} \int_0^1 \|\nabla f(\varphi)\| dt\, dx \\
&= \|P - Q\|^{1-\frac{m}{p}} \int_{B^{m-1}} \int_0^1 \|\nabla f\| \|P - Q\|^{\frac{m}{p}} (t(1-t))^{\frac{m-1}{p}} (t(1-t))^{-\frac{m-1}{p}}\, dt\, dx \\
&\leq \sqrt{2}\|P - Q\|^{1-\frac{m}{p}} \left(\int_{B^{m-1}} \int_0^1 \|\nabla f\|^p J(\varphi)\, dt\, dx \right)^{\frac{1}{p}} \\
&\qquad\qquad \times \left(A(B^{m-1}) \int_0^1 (t(1-t))^{-\frac{m-1}{p-1}}\, dt \right)^{\frac{p-1}{p}} \\
&= \sqrt{2}\|P - Q\|^{1-\frac{m}{p}} \left(\int_{\operatorname{Im}\varphi} \|\nabla f\|^p\, dV \right)^{\frac{1}{p}} \\
&\qquad\qquad \times \left(A(B^{m-1}) B\left(\frac{p-m}{p-1}, \frac{p-m}{p-1} \right) \right)^{\frac{p-1}{p}}
\end{aligned}
$$

As $\operatorname{Im}\varphi \subset B(C, r)$ this implies the inequalities (D.6) and (D.7) and completes the proof. □

Problems

PROBLEM 1. Let M^m be a manifold and g_1, g_2 two Riemannian metrics on M^m. Let V_{g_i} be the volume measure of g_i, A_{g_i} the surface area measure induced on hypersurfaces by g_i, $\nabla_{g_i} f$ the gradient with respect to g_i etc. For $\alpha \geq 1$ show that a "mixed" Sobolev inequality of the type

$$\int_M |f|^\alpha\, dV_{g_1} \leq c_1 \left(\int_M \|\nabla_{g_2} f\|\, dV_{g_2} \right)^\alpha$$

holds if and only if a "mixed" isoperimetric inequality of degree α

$$V_{g_1}(D) \leq c_2 A_{g_2}(\partial D)^\alpha.$$

holds for all $D \subset\subset M$. What is the relationship between the sharp constants in the two inequalities? (The next problem will show that this problem is not as pointless as it may seem.)

PROBLEM 2. Let M be an open set in $\mathbf{R}^m$ and w_1, w_2 positive C^1 functions defined on M. In analysis weighted Sobolev inequalities of the type

$$\left(\int_M |f|^q w_1\, dV\right)^{\frac{1}{q}} \le c\left(\int_M \|\nabla f\|^p w_2\, dV\right)^{\frac{1}{p}}$$

are important. While the theory here does not say much about this problem when $p > 1$ in the case $p = 1$ use the last theorem to give necessary and sufficient conditions for the inequality

$$\left(\int_M |f|^q w_1\, dV\right)^{\frac{1}{q}} \le c\int_M \|\nabla f\| w_2\, dV$$

to hold for all $f \in C_0^\infty(M)$. *Hint*: Consider metrics conformal g_i conformal to the standard flat metric g, that is g_i of the form $g_i = v_i g$. Your final condition should not make any explicit mention of the metrics g_i.

PROBLEM 3. For a domain D in the plane with area A and ∂D of length L the isoperimetric inequality is $4\pi A \le L^2$. There is a generalization of this, due to Banchoff and Pohl, to closed curves with self intersections. Let $c : [0, L] \to \mathbf{R}^2$ be a C^1 unit speed curve with $c(0) = c(L)$. For any point $P \in \mathbf{R}^2$ let $w_c(P)$ be the winding number of c about P. Then the Banchoff–Pohl inequality is

$$4\pi \int_{\mathbf{R}^2} w_c(P)^2\, dA(P) \le L^2.$$

Prove this inequality from the Sobolev inequality $4\pi \int |f|^2\, dA \le \left(\int \|\nabla f\|\, dA\right)^2$ which holds for all $f \in C_0^\infty(\mathbf{R}^2)$.

PROBLEM 4. A subset A of $\mathbf{R}^m$ has width $\le w$ if and only if is there are orthonormal coordinates $x^1, \dots, x^m$ on $\mathbf{R}^m$ so that A is contained in the set defined by $-w/2 \le x^1 \le w/2$. Let M be an open subset of $\mathbf{R}^m$ of width $\le w$. Then show for any $D \subset\subset M$ that

$$V(D) \le \frac{w}{2} A(\partial D)$$

and that this inequality is sharp. This shows that the isoperimetric constant $h_1(M) \le w/2$ for any domain of width $\le w$. *Remark*: The Cheeger inequality then implies that the first eigenvalue of M satisfies $\lambda_1(M) \ge 1/w^2$. The sharp inequality is $\lambda_1(M) \ge \pi^2/w^2$.

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Index