THE GRONWALL INEQUALITY

RALPH HOWARD
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, S.C. 29208, USA
HOWARD@MATH.SC.EDU

1. Introduction.

The Gronwall inequality as given here estimates the difference of solutions to two differential equations $y'(t) = f(t, y(t))$ and $z'(t) = g(t, z(t))$ in terms of the difference between the initial conditions for the equations and the difference between f and g. The usual version of the inequality is when $f = g$, but there is no extra work involved in proving the more general case. The original inequality seems to have first appeared in 1919 in a paper [1] of Gronwall. These notes are based on a lecture and some homework problems given in a graduate class in ordinary differential equations in the spring of 1997.

2. The Inequality

Theorem 2.1 (The Gronwall Inequality)**.** *Let* $\mathbf{X}$ *be a Banach space and* $U \subset \mathbf{X}$ *an open set in* $\mathbf{X}$*. Let* $f, g\colon [a, b] \times U \to \mathbf{X}$ *be continuous functions and let* $y, z\colon [a, b] \to U$ *satisfy the initial value problems*

$$y'(t) = f(t, y(t)), \quad y(a) = y_0, \tag{2.1}$$

$$z'(t) = g(t, z(t)), \quad z(a) = z_0. \tag{2.2}$$

Also assume there is a constant $C \geq 0$ *so that*

$$\|g(t, x_2) - g(t, x_1)\| \leq C\|x_2 - x_1\| \tag{2.3}$$

and a continuous function $\varphi\colon [a, b] \to [0, \infty)$ *so that*

$$\|f(t, y(t)) - g(t, y(t))\| \leq \varphi(t). \tag{2.4}$$

Then for $t \in [a, b]$

$$\|y(t) - z(t)\| \leq e^{C|t-a|}\|y_0 - z_0\| + e^{C|t-a|} \int_a^t e^{-C|s-a|}\varphi(s)\, ds. \tag{2.5}$$

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Remark 2.2. The inequality (2.4) is a little awkward as it involves an inequality along the solution $y(t)$ which we may not know. But we can replace (2.4) by the stronger hypothesis

$$\|f(t,x) - g(t,x)\| \le \varphi(t) \quad \text{for all} \quad x \in U \tag{2.6}$$

(which clearly implies (2.4)) and get the same result. Of course this may mean using a choice of φ that is larger than is needed in (2.4). □

Remark 2.3. Note we are not assuming that f satisfies a Lipschitz condition and therefore solutions to the initial value problem $y'(t) = f(t, y(t))$, $y(a) = y_0$ need not be unique. In this case the inequality can be used to estimate $y(t)$ by comparing it to the solution to $z'(t) = g(t, z(t))$, $z(a) = y_0$ with the same initial condition. In this case the inequality becomes

$$\|y(t) - z(t)\| \le e^{C|t-a|} \int_a^t e^{-C|s-a|} \varphi(s)\, ds.$$

Then if we assume that the inequality (2.6) holds then and $y_1'(t) = f(t, y_1(t))$ with $y_1(a) = y_1$ then using the obvious inequality $\|y(t) - y_1(t)\| \le \|y(t) - z(t)\| + \|z(t) - y_1(t)\|$ where $z'(t) = g(t, z(t))$, $z(a) = y_0$ we get the inequality

$$\|y(t) - y_1(t)\| \le e^{C|t-a|}\|y_0 - y_1\| + 2e^{C|t-a|} \int_a^t e^{-C|s-a|} \varphi(s)\, ds.$$

This gives a version of the Gronwall inequality for differential equations that do not satisfy a Lipschitz condition in terms of the Lipschitz constant of a "nearby" differential equation that does (where how near is "nearby" is measured by the size of φ). □

Proof of Theorem 2.1. We will use the inequality $\frac{d}{dt}\|x(t)\| \le \|x'(t)\|$ which is easily shown to hold for C^1 functions $x\colon [a,b] \to \mathbf{X}$. Then, using the assumptions (2.4) and (2.3),

$$\begin{aligned}
\frac{d}{dt}\|y(t) - z(t)\| &\le \|y'(t) - z'(t)\| \\
&= \|f(t, y(t)) - g(t, z(t))\| \\
&\le \|f(t, y(t)) - g(t, y(t))\| + \|g(t, y(t)) - g(t, z(t))\| \\
&\le \varphi(t) + C\|y(t) - z(t)\|.
\end{aligned}$$

That is

$$\frac{d}{dt}\|y(t) - z(t)\| - C\|y(t) - z(t)\| \le \varphi(t).$$

Multiplication by the integrating factor e^{-Ct} yields

$$\frac{d}{dt}\left(e^{-Ct}\|y(t) - z(t)\|\right) \le e^{-Ct}\varphi(t).$$

Integrate this from a to t to get

$$e^{-Ct}\|y(t) - z(t)\| - e^{-Ca}\|y_0 - z_0\| \le \int_a^t e^{-Cs}\varphi(s)\, ds.$$

This is equivalent to (2.5). □

The following is the standard form of the Gronwall inequality.

Corollary 2.4. *Let* $\mathbf{X}$ *be a Banach space and* $U \subset \mathbf{X}$ *an open set in* $\mathbf{X}$. *Let* $f\colon [a,b] \times U \to \mathbf{X}$ *be a continuous function and let* $y, z\colon [a,b] \to U$ *satisfy the initial value problems*

$$y'(t) = f(t, y(t)), \quad y(a) = y_0, \tag{2.7}$$

$$z'(t) = f(t, z(t)), \quad z(b) = z_0. \tag{2.8}$$

Also assume there is a constant $C \geq 0$ *so that*

$$\|f(t,x_2) - f(t,x_1)\| \leq C\|x_2 - x_1\|. \tag{2.9}$$

Then for $t \in [a,b]$

$$\|y(t) - z(t)\| \leq e^{C|t-a|}\|y_0 - z_0\|. \tag{2.10}$$

Proof. In Theorem 2.1 let $f = g$. Then we can take $\varphi(t) \equiv 0$ in (2.4). Then (2.5) reduces to (2.10). □

3. The Gronwall Inequality for Higher Order Equations

The results above apply to first order systems. Here we indicate, in the form of exercises, how the inequality for higher order equations can be reduced to this case.

Consider the n^{th} order scalar initial value problem

$$\begin{gathered} y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t)), \\ y(0) = y_0,\ y'(0) = y_1,\ \ldots,\ y^{(n-1)}(0) = y_{n-1}. \end{gathered} \tag{3.1}$$

We can transform this into a single first order vector equation by letting

$$Y(t) := \begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$$

and for a vector $X = [x_0, x_1, \ldots, x_{n-1}]^t$ (there A^t is the transpose of A).

$$F(t,X) := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ f(t, x_0, \ldots, x_{n-1}) \end{bmatrix}$$

Set $Y_0 := [y_0, y_1, \ldots, y_{n-1}]^t$. Then consider the first order intial value problem for the vector $Y(t)$.

$$Y'(t) = F(t, Y(t)), \quad Y'(0) = Y_0. \tag{3.2}$$

Exercise 3.1. (1) Give a brief indication why the scalar initial value problem (3.1) and the vector initial value problem (3.2) are equivalent.

(2) Take the version Gronwall's inequality given in Theorem 2.1 and use it to give a version of Gronwall's inequality for the higher order scalar equation (3.1). (This will involve two n^{th} scaler equations as Theorem 2.1 involves two first order systems.)

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References

1. T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations.*, Ann. of Math. **20** (1919), 292–296, 4.