GENERIC CUT LOCI ARE DENSE

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Abstract. Let $\mathcal{K}$ be the complete metric space of compact subsets of $\mathbf{R}^n$ with the Hausdorff distance. For $K \in \mathcal{K}$ let $\mathrm{Cut}(K)$ be the cut locus of K in $\mathbf{R}^n$. We show that $\mathcal{D} := \{K \in \mathcal{K} : \mathrm{Cut}(K) \text{ is dense in } \mathbf{R}^n \setminus K\}$ is a dense G_δ in $\mathcal{K}$.

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Contents

1. Introduction

Let $A \subset \mathbf{R}^n$ be a compact set and let

$$\rho_A(x) := \min\{\|x - a\| : a \in A\}$$

be the distance from A. We are interested in studying the regularity properties (such as where is is differentiable) in terms of geometric properties of A. Much of the regularity of ρ_A can be understood in terms of two geometric concepts: generators and cut points. For an interval I a unit speed curve $c\colon I \to \mathbf{R}^n$ is ***A-minimizing*** iff $\rho_A(c(t)) = t$ for all $t \in I$. An A-minimizing segment with maximal domain is called an ***A-generator***. When A is clear from context we will just refer to generators. We will show that every A-minimizing segment is contained in an A-generator and that the domain of an A-generator is either of the form $[0, L]$ (when the generator has finite length) or $[0, \infty)$ (when the generator has infinite length). If $c\colon [0, L] \to \mathbf{R}^n$ is a finite length A-generator, then its endpoint $c(L)$ is the ***cut point*** of c. The set

$$\mathrm{Cut}(A) = \{c(L) : \ c\colon [0, L] \to \mathbf{R}^n \text{ is a finite length } A\text{-genator.}\}$$

is the ***cut locus*** of A. The elements of $\mathrm{Cut}(A)$ are called cut point of A. The relation of these concepts to regularity properties of ρ_A is that for $x \in \mathbf{R}^n \setminus A$ the derivative $d\rho_A(x)$ exists at x if and only if x is on exactly one generator of A (Theorem 6.8) and the set of non-differentiable points of ρ_A is dense in $\mathrm{Cut}(A)$ (Proposition 6.12).

If A is an embedded C^2 submanifold of $\mathbf{R}^n$ then it is well known that $\mathrm{Cut}(A)$ is a closed subset of $\mathbf{R}^n$. However for more general sets it is known that $\mathrm{Cut}(A)$ need not be closed and that it can be dense in an open set (cf. [7, 5]). We will show that this later behavior is generic. To make this precise, let $\mathcal{K}$ be the collection of nonempty compact subsets of $\mathbf{R}^n$ and $d_H(A, B)$ be the Hausdorff distance between A and B (see Definition 2.1 for the precise definition). Then $(\mathcal{K}, d_H)$ is a complete metric space (Proposition 2.2).

Let

$$\mathcal{D} := \{A \in \mathcal{K} : \mathrm{Cut}(A) \text{ is a dense subset of } \mathbf{R}^n \setminus A\}.$$

Our main result is that almost all, is the sense of Baire category, elements of $\mathcal{K}$ are in $\mathcal{D}$. Recall that a set in a metric space is a G_δ iff it is a countable intersection of open sets.

Theorem 1. *The set $\mathcal{D}$ is a dense G_δ in $\mathcal{K}$.*

It will be shown (Theorem 6.13) that $\mathcal{D}$ is also the set of A so that ρ_A is not differentiable on any open subset of $\mathbf{R}^n \setminus A$. Theorems 1 and 6.13 are closely related to [5, Thm. 3 p. 5141] where they prove a related result with $\mathcal{K}$ replaced by the compact convex sets. These results show that for generic $A \in \mathcal{K}$ the function ρ_A is nowhere C^1. The proof here is an adaption of Banach's [3] well known proof that almost every, in the sense of Baire category, function in $C[0, 1]$ is nowhere differentiable. (For a very nice presentation of Banach's proof and other ideas related to the Baire category theorem see the readable and informative little book [11] of Oxtoby.)

While the results here are stated and proven for compact subsets of Euclidean space $\mathbf{R}^n$, it is not hard to see that they hold with only slightly rewritten proofs for compact subsets of a complete Riemannian manifold (and with some extra work and appropriate definitions in incomplete Riemannian manifolds or even smooth Finsler manifolds).

2. The Hausdorff distance

We start by making the collection of compact subsets of a complete metric space into another metric space.

2.1. **Definition** (Hausdorff 1914 [10]). Let (X, d) be a complete metric space and $\mathcal{K}$ the collection of all nonempty compact subsets of X. Define a distance d_H on $\mathcal{K}$ by

$$\begin{aligned} d_H(A, B) &:= \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\} \\ &= \inf \left\{ r : \begin{array}{l} \text{Every point of } A \text{ is within } r \text{ of a point of } B \text{ and} \\ \text{every point of } B \text{ is within } r \text{ of a point of } A. \end{array} \right\} \end{aligned}$$

Then d_H is the ***Hausdorff distance*** on $\mathcal{K}$. $\square$

2.2. **Proposition** (Hausdorff 1914 [10]). *If (X, d) is a compete metric space, then $(\mathcal{K}, d_H)$ is also a complete metric space.*

Proof. To start:

Exercise 1. The function d_H satisfies the triangle inequality on $\mathcal{K}$ and if $d_H(A, B) = 0$, then $A = B$. $\square$

This shows that (X, d_H) is a metric space. To show completeness let $\{A_\ell\}_{\ell=1}^\infty$ is a Cauchy sequence in $(\mathcal{K}, d_H)$ and let A be the set of all points a that are limits of convergent sequences $\{a_\ell\}_{\ell=1}^\infty$ with $a_\ell \in A_\ell$. From its definition A is easily seen to be closed and bounded. If (X, d) is a space where closed bounded sets are compact (such as $\mathbf{R}^n$) then we have that A is compact without any more work.

But if (X, d) has closed bounded sets that are not compact (Hilbert space is such an example) more work is needed to show A is compact. This will be based on:

Exercise 2. Let (Y, d) be a metric space. Show (Y, d) is compact if and only if (Y, d) is complete and for every ε there is a finite number of open ε balls $\{B(y_i, \varepsilon)\}_{i=1}^N$ that cover Y. (A metric space that can be covered by a finite number of ε balls for any ε is called ***totally bounded***. Therefore a metric space is compact if and only if it is complete and totally bounded.) $\square$

We next show that the Hausdorff distance between A and the terms in $\{A_\ell\}_{\ell=1}^\infty$ goes to zero.

Exercise 3. Let $\varepsilon > 0$ and choose that ℓ so that if $m, n \geq \ell$, then $d_H(A_m, A_n) < \varepsilon$. Then show that if $n \geq \ell$ then for any point a of A_n there is a point b of A with $d(a, b) \leq \varepsilon$ and for every point b of A there is a point a of A_n with $d(a, b) \leq \varepsilon$. $\square$

Returning to our set $A \subset X$, it is a closed subset of compete metric space (X, d) and therefore A is complete as a metric space. So we only need show that it is totally bounded to show that it is compact. Let $\varepsilon > 0$. Then choose ℓ so that if $m, n \geq \ell$, then $d_H(A_m, A_n) < \varepsilon/3$. As A_ℓ is compact there is a finite subset $\{b_1, \ldots, b_N\} \subset A_\ell$ so that every point of A_ℓ is at a distance $< \varepsilon/3$ from one of the b_i's. By Exercise 3 for each b_i there is an $a_i \in A$ with $d(a_i, b_i) \leq \varepsilon/3$. Now let $a \in A$. There there is a point $b \in A_\ell$ with $d(a, b) \leq \varepsilon/3$ and a b_i with $d(b, b_i) < \varepsilon/3$. Therefore

$$d(a, a_i) \leq d(a, b) + d(b, b_i) + d(b_i, a_i) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus $A \subset \bigcup_{i=1}^N B(a_i, \varepsilon)$ and so A is totally bounded. Therefore A is compact as claimed.

Finally $d_H(A, A_\ell) \to 0$ follows from Exercise 3. So every Cauchy sequence in $(\mathcal{K}, d_H)$ converges and therefore it is complete. $\square$

For a nonempty subset A of the a metric space (X,d) let

$$\rho_A(x) := \inf\{d(x,a) : a \in A\}. \tag{2.1}$$

If A is compact then the infimum can be replaced by a minimum

$$\rho_A(x) := \min\{d(x,a) : a \in A\} \quad \text{when } A \text{ is compact.}$$

The triangle inequality implies

$$\rho_A(y) \le \rho_A(x) + d(x,y). \tag{2.2}$$

Interchanging the rôles of x and y gives $\rho_A(x) \le \rho_A(y) + d(y,x)$ and so

$$|\rho_A(x) - \rho_A(y)| \le d(x,y).$$

If $f\colon X \to \mathbf{R}$ is continuous then the sup norm of f is

$$\|f\|_{L^\infty} = \sup_{x\in X} |f(x)|.$$

2.3. **Proposition.** *If (X,d) is a compete metric space the Hausdorff distance on $(\mathcal{K}, d_H)$ is given by*

$$d_H(A,B) = \|\rho_A - \rho_B\|_{L^\infty}$$

where ρ_A is defined by (2.1).

Proof. Let $a \in A$ and let $b \in B$ be so that $d(a,b) = \rho_B(a)$. Then, using $\rho_A(a) = 0$, we have

$$d(a,b) = \rho_B(a) - \rho_A(a) = |\rho_B(a) - \rho_A(a)| \le \|\rho_A - \rho_B\|_{L^\infty}.$$

Thus we have shown that for any $a \in A$ there is a $b \in B$ with $d(a,b) \le \|\rho_A - \rho_B\|_{L^\infty}$. By symmetry there for each $b \in A$ there is an $a \in A$ with $d(a,b) \le \|\rho_A - \rho_B\|_{L^\infty}$. Therefore $d_H(A,B) \le \|\rho_A - \rho_B\|_{L^\infty}$.

To finish the proof it is enough to show $\|\rho_A - \rho_B\|_{L^\infty} \le d_H(A,B)$. If $x \in X$ let $a \in A$ be so that $d(x,a) = \rho_A(x)$. Then there is a $b \in B$ with $d(a,b) \le d_H(A,B)$. Therefore

$$\rho_B(x) \le d(x,b) \le d(x,a) + d(a,b) = \rho_A(x) + d(a,b) \le \rho_A(x) + d_H(A,B).$$

That is $\rho_B(x) - \rho_A(x) \le d_H(A,B)$. Interchanging the rôles of A and B gives $\rho_A(x) - \rho_B(x) \le d_H(A,B)$. Therefore $|\rho_B(x) - \rho_A(x)| \le d_H(A,B)$. Whence

$$\|\rho_B - \rho_A\| = \sup_{x\in X} |\rho_B(x) - \rho_A(x)| \le d_H(A,B),$$

which completes the proof. $\square$

2.4. **Corollary.** *If $A_\ell, A \in \mathcal{K}$ and $d(A, A_\ell) \to 0$, then $\rho_{A_\ell} \to 0$ uniformly. If also $c_\ell, c\colon [a,b] \to X$ are continuous maps and $c_\ell \to c$ uniformly then $\rho_{A_\ell}(c_\ell(t)) \to \rho_A(c(t))$ uniformly.*

Proof. This follows at once from the last proposition and general results about uniform convergence. $\square$

3. Minimizing segments and generators

We first show that minimizing segments really are parts of line segments.

3.1. **Proposition.** *Let $A \in \mathcal{K}$ and let $c\colon [a,b] \to \mathbf{R}^n$ be an A-minimizing segment. Then c is a unit speed parameterization of the line segment between $c(a)$ and $c(b)$. Explicitly:*

$$c(t) = \frac{1}{\|c(b) - c(a)\|}\big((b-t)c(a) + (t-a)c(b)\big)$$

Proof. The triangle inequality implies that $|\rho_A(x) - \rho_A(y)| \leq \|x - y\|$ for all $x, y \in \mathbf{R}^n$. As an A-minimizing segment $c\colon [a,b] \to \mathbf{R}^n$ is unit speed we then have the length of c is $b-a$ is $\geq$ the distance $\|c(b) - c(a)\|$ between the endpoints of c. Therefore

$$\|c(b) - c(a)\| \leq \operatorname{Length}(c) = (b-a) = \rho_A(c(b)) - \rho_A(c(a)) \leq \|c(b) - c(a)\|.$$

Thus c minimizes the distance between its endpoints and whence is a parameterization of a line segment. $\square$

3.2. **Proposition.** *Let $A \in \mathcal{K}$. Then every $x \in \mathbf{R}^n \smallsetminus A$ is on a A-generator. The domain of an A-generator c is either $[0, L]$ for some $L > 0$ or of the from $[0, \infty)$ and in either case $c(0) \in A$.*

Proof. If $x \notin A$, then, as A is compact, there is a point $z \in A$ so that $\|x - z\| = \rho_A(x)$. Then $c(t) = \|x - z\|^{-1}((1-t)z + tx)$ with $t \in [0, \rho_A(x)]$ is an A-minimizing segment. Thus every point is on at least one such segment. For the segment $c\colon [0, \rho_A(x)] \to \mathbf{R}^n$ just defined let $\overline{c}\colon [0,\infty) \to \mathbf{R}^n$ be the natural extension of c to $[0, \infty)$ (that is $\overline{c}(t) = \|x - z\|^{-1}((1-t)z + tx)$ for all $t \geq 0$). Let

$$L := \sup\{b : \overline{c}\big|_{[0,b]} \text{ is an } A\text{-minimizing segment.}\}$$

If $L < \infty$ then $\overline{c}\big|_{[0,L]}\colon [0, L] \to \mathbf{R}^n$ is an A-generator. If $L = \infty$ then $\overline{c}\colon [0, \infty) \to \mathbf{R}^n$ is an A-generator. From the construction of $\overline{c}$ as an extension of c we have that $L \geq \rho_A(x)$ and therefore x is on the generator just constructed.

If $c\colon [a, b] \to \mathbf{R}^n$ is an A-minimizing segment with $a > 0$, then let $z \in A$ be a point with $\|c(a) - z\| = a$ (note $a = \rho_A(c(a))$ by definition of A-minimizing) and define $c_0\colon [0, b] \to \mathbf{R}^n$ by

$$c_0(t) := \begin{cases} a^{-1}((1-t)z + tx), & t \in [0, a]; \\ c(t), & t \in [a, b]. \end{cases}$$

This is an A-minimizing segment. Thus every A-minimizing segment defined on an interval $[a, b]$ can be extended to an A-minimizing segment on an interval $[0, b]$. It then follows that A-minimizing segments with maximal domain are either of the for $c\colon [0, L] \to \mathbf{R}^n$, or of the form $c\colon [0, \infty) \to \mathbf{R}^n$. Finally if $c(0)$ will have distance zero from A and therefore $c(0) \in A$ as A is closed. $\square$

3.3. **Proposition.** *If $A \in \mathcal{K}$ and $z \in \mathbf{R}^n \smallsetminus A$ is on two A-generators, then z is a cut point of both generators.*

Proof. Let z be on both c_1 and c_2, which we assume are distinct. Let $a = \rho_A(z)$. Then $z = c_1(a) = c_2(a)$. If z is not a cut point of one of the two segments, say c_2, then we have for some $b > a$ that $c_2\big|_{[0,b]}$ is an A-minimizing segment. Let $c\colon [0,b] \to \mathbf{R}^n$ be defined by

$$c(t) = \begin{cases} c_1(t), & t \in [0,a]; \\ c_2(t), & t \in [a,b]. \end{cases}$$

Then c will be A-minimizing and therefore by Proposition 3.1 a line segment. But this implies that $c_1 = c_2$ contrary to the assumption that c_1 and c_2 are distinct. □

3.4. **Proposition.** *Let $U \subset \mathbf{R}^n$ be a non-empty open set. Then*

$$\mathcal{K}_U := \{A \in \mathcal{K} : U \cap (A \cup \operatorname{Cut}(A)) = \varnothing\}$$

is a closed nowhere dense subset of $\mathcal{K}$.

Proof. Let $A_\ell \in \mathcal{K}_U$ with $\lim_{\ell\to\infty} A_\ell = A$ in $\mathcal{K}$. Then to show that $\mathcal{K}_U$ is closed we need that $A \in \mathcal{K}_U$. As $A_\ell \cap U = \varnothing$ for all ℓ we have $A \cap U = \varnothing$. Let $z \in U$ and $2r = \operatorname{dist}(z, \partial U)$. For each ℓ let $a_\ell = \rho_{A_\ell}(z)$. Then there is an A_ℓ-generator c_ℓ with $c_\ell(a_\ell) = z$ and, as c_ℓ has no endpoint in U, the domain of c_ℓ is at least $[0, a_\ell + 2r]$. Let $a = \rho_A(z)$. Then $\lim_{\ell\to\infty} a_\ell = a$ and by going to a subsequence we can assume that $\{c_\ell\}_{\ell=1}^\infty$ converges uniformly to some segment c on $[0, a+r]$. But then for $t \in [0, a+r]$ we have

$$\rho_A(c(t)) = \lim_{\ell\to\infty} \rho_{A_\ell}(c_\ell(t)) = \lim_{\ell\to\infty} t = t$$

and therefore $c\colon [0, a+r] \to \mathbf{R}^n$ is an A-minimizing segment. By Proposition 3.3 $z = c(a)$ can not be on any other A-minimizing segment (otherwise it would be a cut point of c, contradicting that $c(a)$ is in the interior of $c\colon [0, a+r] \to \mathbf{R}$). Therefore $z = c(a)$ is not a cut point of any A-minimizing segment and so $z \notin \operatorname{Cut}(A)$. As z was any point of U this shows that $U \cap (A \cup \operatorname{Cut}(A)) = \varnothing$ and completes the proof that $\mathcal{K}_U$ is closed.

To show that $\mathcal{K}_U$ is nowhere dense let $\varepsilon > 0$ and $A \in \mathcal{K}_U$. Let $z \in U$. Then by definition of $\mathcal{K}_U$, z is not a cut point of A. Let $a = \rho_A(z)$ and let $c\colon [0,b] \to \mathbf{R}^n$ be a minimizing segment from A to z. Then $c(a) = z$. The closed ball $\overline{B}(z,a)$ meets A in just the one point $c(0) \in A$, for otherwise there would be two minimizing segments from z to A, which would imply z is a cut point of A (by Proposition 3.3). Therefore we can choose a point x on the sphere $S(z,a) := \partial B(z,a)$ with $x \neq c(0)$ and $\|x - c(0)\| < \varepsilon$. Let $A_1 = A \cup \{x\}$. Then

$$d_H(A, A_1) = \operatorname{dist}(x, A) \leq \|x - c(0)\| < \varepsilon.$$

Also $\operatorname{dist}(a, A_1) = a$ and $\overline{B}(z,a) \cap A = \{c(0), z\}$ so that there will be two minimizing segments from A_1 to z. Therefore z is a cut point of A. This shows that for all $\varepsilon > 0$ and any $A \in \mathcal{K}_U$ there is an $A_1 \notin \mathcal{K}_U$ with $d(A, A_1) < \varepsilon$. Thus $\mathcal{K}_U$ is nowhere dense. □

4. The Baire Category Theorem

The statement of Baire's theorem which is easiest to prove is:

4.1. **Theorem** (Baire Category Theorem, Baire 1899 [2]). *Let (X, d) be a complete metric space. Then the intersection of a countable collection of dense open sets is dense.*

Exercise 4. Prove this. Hint: Let the collection of dense open sets be $\{U_\ell\}_{\ell=1}^\infty$. By replacing U_ℓ with the intersection $U_1 \cap U_2 \cap \cdots \cap U_\ell$ we can assume that $U_{\ell+1} \subseteq U_\ell$ for all ℓ. Let $B(a, r)$ be the open ball of radius r about a. Then to show that $\bigcap_{\ell=1}^\infty U_\ell$ is dense, it is enough to show that it meets each $B(a, r)$. As U_1 is open and dense the set $B(a, r) \cap U_1$ is open and non-empty. So there is closed ball $\overline{B}(a_1, r_1) \subset U_1 \cap B(a, r)$ and we can arrange that $r_1 \leq 1$. Likewise $U_2 \cap B(a_1, r_1)$ is non-empty and open so there is a closed ball $\overline{B}(a_2, r_2) \subset U_2 \cap B(a_1, r_1)$ and we can choose this so that $r_2 \leq 1/2$. Continuing in this manner we get a sequence $\{\overline{B}(a_\ell, r_\ell)\}$ with $\overline{B}(a_\ell, r_\ell) \subset U_\ell \cap B(a_{\ell-1}, r_{\ell-1})$ and $r_\ell \leq 1/\ell$. Now show that $\{a_\ell\}_{\ell=1}^\infty$ is Cauchy and use completeness to show $B(a, r) \cap \bigcap_{\ell=1}^\infty U_\ell \supseteq \bigcap_{\ell=1}^\infty \overline{B}(a_\ell, r_\ell) \neq \varnothing$. □

As the compliment of dense open set is a closed nowhere dense set and the compliment of an intersection is a union we can dualize the form of Baire's theorem above.

4.2. **Theorem** (Baire Category Theorem Second Form). *Let (X, d) be a complete metric space. Then the union of a countable collection of nowhere dense subsets of X has dense compliment in X. In particular X is not the a countable union of nowhere dense subsets.*

Proof. Let $\{F_\ell\}_{\ell=1}^\infty$ be a countable collection of nowhere dense subsets of X. Then $U_\ell := X \setminus \overline{F}_\ell$ is open and dense. Therefore $X \setminus \bigcup_{\ell=1}^\infty F_\ell \supseteq X \setminus \bigcup_{\ell=1}^\infty \overline{F}_\ell = \bigcap_{\ell=1}^\infty U_\ell$ is dense. □

4.3. **Definition.** Let (X, d) be a metric space and $A \subseteq X$. Then A is ***meager*** iff A is contained in a countable union of closed nowhere dense sets. A set is ***comeager*** iff it is the compliment of a meager set. (Equivalently A is comeager iff it contains a countable intersection of dense open sets.) □

These definitions imply

4.4. **Proposition.** *A countable union of meager sets is meager. A countable intersection of comeager sets is comeager.* □

Note that if (X, d) is a complete metric space, no set can be both meager and comeager. For if A was such a set, then $X \setminus A$ would also be meager and therefore $X = A \cup (X \setminus A)$ would be meager. But this contradicts

Theorem 4.2 which implies that X is not a countable union of nowhere dense sets.

The next proposition is not important in applications of the Baire Theorem, but it does formalize, at least a bit, the idea that meager subsets of X are "small" and the comeager sets are "large".

4.5**. Proposition.** *Let (X, d) be a complete metric space. Let $\mathcal{C}$ be the set of all subsets of X that are either meager or comeager. Then $\mathcal{C}$ is a σ-algebra and if $\mu\colon \mathcal{C} \to \mathbf{R}$ is defined by*

$$\mu(A) = \begin{cases} 0, & \textit{if } A \textit{ is meager;} \\ 1, & \textit{if } A \textit{ is comeager.} \end{cases}$$

then μ is a measure on $\mathcal{C}$.

Exercise 5. Prove this. Hint: To show countably additivity of μ note that any two comeager sets intersect non-trivially, so if $\{A_\ell\}_{\ell=1}^{\infty}$ is a pairwise disjoint sequence from $\mathcal{C}$ then at most one of the A_ℓ's is comeager. Therefore $\sum_{\ell=1}^{\infty} \mu(A_\ell)$ is 0 if all the A_ℓ's are meager, and is 1 if one of the A_ℓ's is comeager. □

Motivated by Proposition 4.5 if $P(x)$ is a property of points of X so that $\{x \in X : P(x)\}$ is comeager, then is often said that "$P(x)$ holds for almost all $x \in X$ in the sense of category". A more common terminology is that "the generic point $x \in X$ has property $P(x)$". In this setting "generic" means "holds on a comeager set". For example the generic real number is irrational.

5. Proof of Theorem 1

We will prove the dual of theorem. Let $\mathcal{K}_U$ be as in the statement of Proposition 3.4. That is $\mathcal{K}_U$ is the collection of $A \in \mathcal{K}$ so that $U \cap (A \cup \operatorname{Cut}(A)) = \varnothing$. Set

$$\mathcal{R} := \bigcup \{\mathcal{K}_U : \ U \text{ is a non-empty open subset of } \mathbf{R}^n\}.$$

Then $\mathcal{R}$ is exactly the set of all A so that both A and $\operatorname{Cut}(A)$ are disjoint from some nonempty open set. That is $\mathcal{R} = \mathcal{K} \smallsetminus \mathcal{D}$. Therefore Theorem 1 can be restated as

5.1**. Theorem.** *The set $\mathcal{R}$ is a countable union of closed nowhere dense subsets of $\mathcal{K}$. (Thus by Theorem 4.2 $\mathcal{R}$ is meager in $\mathcal{K}$ and therefore $\mathcal{D}$ is comeager.)*

Proof. Let $\mathcal{B}$ the collection of open balls $B(a, r)$ in $\mathbf{R}^n$ where all the components of a and the number r are rational. This is a countable set and every open subset of $\mathbf{R}^n$ is a countable union of elements of $\mathcal{B}$. By Proposition 3.4 the set $\mathcal{K}_B$ is a closed nowhere dense subset of $\mathcal{K}$ and thus $\bigcup\{\mathcal{K}_B : B \in \mathcal{B}\}$ is a countable union of closed nowhere subsets of $\mathcal{K}$.

To finish it is enough to show $\bigcup\{\mathcal{K}_B : B \in \mathcal{B}\} = \mathcal{R}$. Clearly $\bigcup\{\mathcal{K}_B : B \in \mathcal{B}\} \subseteq \mathcal{R}$. If $A \in \mathcal{R}$, then there is a nonempty open subset U of $\mathbf{R}^n$ so that $A \in \mathcal{K}_U$. There will be a nonempty open ball $B \in \mathcal{B}$ with $B \subset U$. But then $A \in \mathcal{K}_B$ and thus $A \in \bigcup\{\mathcal{K}_B : B \in \mathcal{B}\}$. Therefore $\mathcal{R} \subseteq \bigcup\{\mathcal{K}_B : B \in \mathcal{B}\}$. □

6. The cut locus and regularity properties of ρ_A

6.1. **Definition.** Let U be an open subset of $\mathbf{R}^n$ and $f\colon U \to \mathbf{R}^n$. Then f is ***semi-concave*** iff every point of U has an open convex neighborhood V so that $f\big|_V = h + \varphi$ where h is concave and φ is C^∞. □

6.2. **Definition.** Let U be an open subset of $\mathbf{R}^n$ and $f\colon U \to \mathbf{R}^n$. Then a function φ defined in an neighborhood of $x_0 \in U$ is an ***upper support function*** for f at x_0 iff $\varphi(x_0) = f(x_0)$ and $f \le \varphi$ in a neighborhood of x_0. A ***lower support function*** for f at x_0 is defined similarly. □

If φ is a C^2 function with Hessian $D^2\varphi$ (which we view as a symmetric bilinear form) then we say hat $D^2\varphi(x_0) \ge C$ iff $D^2\varphi(x_0)(v,v) \ge C\|v\|^2$ for all vectors v. Likewise $D^2\varphi(x_0) \le C$ iff $D^2\varphi(x_0)(v,v) \le C\|v\|^2$ for all vectors v.

6.3. **Proposition.** *Let $U \subseteq \mathbf{R}^n$ be open and convex and let $f\colon U \to \mathbf{R}$ be continuous and such that for each $x_0 \in U$ there is a C^2 upper support function φ to f at x_0 with $D^2\varphi(x_0) \le C$. Then the function $f(x) - \frac{1}{2}C\|x\|^2$ is concave on U. (A function f satisfying the the hypothesis of this proposition is said to* ***satisfy*** $D^2 f \le C$ ***in the sense of support functions****.*

Proof. (*Taken form* [1].) We first give the proof in the one dimensional case. Then $U \subseteq \mathbf{R}$ is an interval. Let φ be an upper support function to f at x_0 that satisfies $D^2\varphi = \varphi'' \le C$ near x_0 and let $a = \varphi'(x_0)$. We claim $f(x) \le f(x_0) + a(x - x_0) - (C/2)(x - x_0)^2$. Let $g(x) := f(x_0) + a(x - x_0) - (C/2)(x - x_0)^2$. As $\varphi(x_0) = g(x_0)$, $\varphi'(x_0) = g'(x_0)$ and $\varphi''(x) \le C = g''(x)$ we have an interval about x_0 so that $f(x) \le \varphi(x) \le g(x)$. Assume, toward a contradiction, there is an x_1 so that $f(x_1) - g(x_1) > 0$. Then the function $g_1 := f - g$ will satisfy $g_1'' \le 0$ in the sense of support functions and $g_1(x_0) = 0$, $g_1(x_1) > 0$ and the function $g_1(x) \le 0$ for x near x_0 so the function g_1 will have an minimum at a point x_* between x_0 and x_1. Let g_0 be the constant function $g_0(x) = v_1(x_*)$. Then $g_1 \ge g_0$, $g_1(x_*) - g_0(x_*) = 0$, and $v_1'' \le 0$ (in the sense of support functions) and $g_0'' = 0$ (in the strong sense) so the one variable case of of the linear maximum principle (which is easy to verify) implies $g_0(x) = g_1(x)$ for x between x_0 and x_1. As $g_1(x_0) = 0$ and g_0 is constant this implies $g_1(x) = g(x) - f(x)$ for all x between x_0 and x_1. This in particular implies $0 = f(x_1) = g(x_1)$ which contradicts the assumption $f(x_1) - g(x_1) > 0$. Therefore $f - g \ge 0$ on all of U. This implies $f(x) - (C/2)\|x\|^2$ has a linear upper support function on U at x_0. As x_0 was any point of U we have that $f(x) - (C/2)\|x\|^2$ is concave. This completes the proof in the one dimensional case.

We return to the general case. Let φ be a upper support function for f at x_0 that satisfies $D^2\varphi \le C$ near x_0 and let $a := D\varphi(x_0)$. For any unit vector $b \in \mathbf{R}^n$ let $f_b(t) = f(x_0 + tb) - t\langle b, a\rangle$. Then a lower support ψ function to v at $x_0 + t_0 b$ that satisfies $D^2\psi \le C$ yields the lower support function $\psi_b(t) := \psi(x_0 + tb)$ to v_b at t_0 that satisfies $\psi_b''(t) \le C$ near t_0. Thus f_b satisfies the one variable version of the result and so $f_b(t) = f(x_0 + tb) \le v(x_0) - t\langle b, a\rangle + (C/2)t^2$. But as Ω is convex every point of Ω can be written as $x = x_0 + tb$ for some t and some unit vector and so the multidimensional case reduces to the one dimensional case and completes the proof. □

If $f\colon \mathbf{R}^n \to \mathbf{R}$ and df exists at x, then let $\nabla f(x)$ be the vector so that

$$df(x)(V) = \langle \nabla f(x), V\rangle \qquad \text{for all vectors } V.$$

That is ∇f is the usual ***gradient*** of f at x.

For the last proposition to be useful we need a good supply of support functions. In geometric problems this can often be done by using distance functions from points.

Exercise 6. Let $a \in \mathbf{R}^n$ and set

$$\rho_a(x) = \|x - a\|.$$

Then the gradient and Hessian of ρ_a are given by

$$\nabla\rho_a(x) = \frac{x-a}{\|x-a\|}, \qquad D^2\rho_a(x)(v,v) = \frac{1}{\|x-a\|}\left(\|v\|^2 - \frac{\langle x-a, v\rangle^2}{\|x-a\|^2}\right)$$

and thus ρ_a satisfies the inequalities

$$0 \le D^2\rho_a(x) \le \frac{1}{\|x-a\|}. \tag{6.1}$$

This shows that $D^2\rho_a$ is informally bounded above on any set that has positive distance from a. □

6.4. **Theorem.** *For any compact set $A \subset \mathbf{R}^n$ the function ρ_A is semi-concave on $U := \mathbf{R}^n \setminus A$.*

Proof. Let B be an open ball with $\rho_A\big|_B$ bounded below. Then for any $x_0 \in B$ there is an $a_0 \in A$ with $\|x_0 - a_0\| = \rho_A(x_0)$. Let $\varphi(x) := \rho_{a_0} = \|x - a_0\|$. Then $\varphi(x_0) = \varphi_A(x_0)$ and for $x \in B$ we have

$$\rho_A(x) = \inf\{\|x - a\| : a \in A\} \le \|x - a_0\| = \varphi(x).$$

Therefore φ is an upper support function for ρ_A at x_0. Also φ is C^∞ on B and as the distance of B from A is positive there are uniform bounds on the Hessian $D^2\varphi(x)$ for $x \in B$ (which only depend on the distance of B form A by the inequalities (6.1)). By Proposition 6.3 this implies that $\rho_A(x) + \frac{1}{2}C\|x\|^2$ is concave on B. As every point of $\mathbf{R}^n \setminus A$ is in a ball of positive distance from A this shows that ρ_A is semi-concave. □

Differentiability of concave functions can be understood in terms of the super-differential at a point: The differentiable points are the points where the super-differential is just one point. We start with the definition.

6.5. **Definition.** Let $f\colon U \to \mathbf{R}$ be a semi-concave function. Then the ***super-differential*** $\partial f(x_0)$ of f at x_0 is

$$\partial f(x_0) := \{d\varphi(x_0) : \varphi \text{ is a } C^2 \text{ upper support function to } f \text{ at } x_0\}$$

where $d\varphi(x_0)$ is the linear functional $d\varphi(x_0)v := \frac{d}{dt}\varphi(x_0+tv)\big|_{t=0}$. □

Exercise 7. If f is concave the usual definition of $\partial f(x_0)$ form convex analysis is

$$\partial f(x_0) = \{\Lambda : \Lambda \text{ is a linear funcional and } f-\Lambda \text{ has a maximum at } x_0\}.$$

Show that for concave f this agrees with Definition 6.5. □

The following is a standard result from convex analysis.

6.6. **Proposition.** *A semi-concave function f is differentiable at x_0 if and only if the super-differential $\partial f(x_0)$ is a singleton. If $\partial f(x_0) = \{\Lambda\}$, then the differential is $df(x_0) = \Lambda$.* □

The following is due to Joe Fu.

6.7. **Proposition** (Fu [9, Lemma 4.2 pp. 1037–1038])**.** *Let $A \subset \mathbf{R}^n$ be compact and let $x_0 \in \mathbf{R}^n \setminus A$ and set*

$$\mathcal{U}_A(x_0) := \left\{ \frac{1}{\|x_0-a\|}(x_0-a) : a\in A, \|x_0-a\| = \rho_A(x_0)\right\}.$$

(That is $\mathcal{U}_A(x_0)$ is the set of unit tangent vectors to A-generators from A to x_0.) Then

$$\partial\rho_A(x_0) = \textit{convex hull of } \{\langle\cdot,u\rangle : u \in \mathcal{U}_A(x_0)\}.$$

□

For $x \in \mathbf{R}^n \setminus A$ set

$$N_A(x) = \text{number of points in } \mathcal{U}_A(x).$$

$N_A(x)$, which is a positive integer or ∞, is the ***multiplicity*** of A at x. It is just the number of A-generators connecting A to x.

Our result now combine our results to give a Euclidean version of a Lorentzian result of Beem and Królak.

6.8. **Theorem** (Beem and Królak [4])**.** *The function ρ_A is differentiable at $x \in \mathbf{R}^n \setminus A$ if and only if $N_A(x) = 1$.*

Proof. By Proposition 6.6, φ_A is differentiable at x if and only if $\partial\rho_A(x)$ is a single point. By Proposition 6.7 $\partial\rho_A(x)$ is the convex hull of $\mathcal{U}_A(x)$ and therefore $\partial\rho_A(x)$ is a singleton if and only if $N_A(x) = 1$. □

6.9. **Proposition** (Beem and Królak [4])**.** *Assume that ρ_A is differentiable on the open set U which is disjoint form A. Then the map $x \mapsto \nabla\rho_A(x)$ is continuous.*

Proof. It is a general fact about concave, and thus also semi-concave, functions that if they are differentiable on an open set, then they are C^1 on the set. However we can give a direct proof in this case. Let $x, x_\ell \in U$ with $\lim_{\ell\to\infty} x_\ell = x$. As ρ_A is differentiable on U we have $N_A(x) = N_A(x_\ell) = 1$ by Theorem 6.8. Therefore there are unique $a, a_\ell \in A$ with $\|x - a\| = \rho_A(x)$ and $\|x_\ell - a_\ell\| = \rho_A(x_\ell)$. As A is compact $\{a_\ell\}_{\ell=1}^\infty$ will have at least one cluster point. If $b \in A$ is a cluster point of $\{a_\ell\}_{\ell=1}^\infty$, then by going to a subsequence we have may assume $a_\ell \to b$. But then

$$\|x - b\| = \lim_{\ell\to\infty} \|x_\ell - a_\ell\| = \lim_{\ell\to\infty} \rho_A(x_\ell) = \rho_A(x).$$

As $a \in A$ is the only point with $\|x - a\| = \rho_A(x)$ this implies that $b = a$ and therefore a is the only cluster point of $\{a_\ell\}_{\ell=1}^\infty$. Thus $\lim_{\ell\to\infty} a_\ell = a$. Now using Propositions 6.6 and 6.7 we have

$$\lim_{\ell\to\infty} \nabla\rho_A(x_\ell) = \lim_{\ell\to\infty} \frac{1}{\|x_\ell - a_\ell\|}(x_\ell - a_\ell) = \frac{1}{\|x - a\|}(x - a) = \nabla\rho_A(x)$$

and therefore $\nabla\rho_A$ is continuous at x as claimed. □

6.10**. Proposition.** *If ρ_A is differentiable on an open set U which is disjoint from A, then* $\mathrm{Cut}(A) \cap U = \varnothing$*. That is every point of U is an interior point of the unique generator that it is on.*

Proof. Let $x_0 \in U$ and let $t_0 = \rho_A(x_0)$. As the vector field $x \mapsto \nabla\rho_A(x)$ is continuous it follows form the Peano existence for ordinary differential equations [8, Thm 1.3 p. 7 and Sec. 1.5 pp. 15–19] that

$$\dot{c}(t) = \nabla\rho_A(c(t)), \qquad c(t_0) = x_0$$

will have a solution on some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. As we only know that $x \mapsto \nabla\rho_A(x)$ is continuous, we do not know, yet, that this solution is unique, only that at least one solution exists. However when $\nabla\rho_A(x)$ exists it is given by $\nabla\rho_A(x) = \|x - a\|^{-1}(x - a)$ for the unique $a \in A$ with $\|x - a\| = \rho_A(x)$ and so it is a unit vector. Therefore c is unit speed curve.

However

$$\frac{d}{dt}\rho_A(c(t)) = \langle \nabla\rho_A(c(t)), \dot{c}(t)\rangle = \langle \nabla\rho_A(c(t)), \nabla\rho_A(c(t))\rangle = 1$$

as $\nabla\rho_A$ is a unit vector when it exists. This implies that on $(t_0 - \varepsilon, t_0 + \varepsilon)$

$$\rho_A(c(t)) = \rho_A(c(t_0)) + (t - t_0) = \rho_A(x_0) + (t - t_0) = t.$$

This implies that c is an A-minimizing segment. As x_0 is in the interior of this segment, when the segment is extended to an A-minimizing x_0 is not an end point of the generator. Therefore x_0 is not a cut point of A. As x_0 as any point of U this shows that $\mathrm{Cut}(A) \cap U = \varnothing$. That any $x \in U$ is only one one A-generator now follows from Proposition 3.3. □

The "irregularity" of a point of $\mathrm{Cut}(A)$ can be measured by the number of generators it is on. Set

$$\mathrm{Cut}^{[k]}(A) = \{x \in \mathrm{Cut}(A) : \dim \text{convex hull}(\mathcal{U}_A(x)) \geq k\}.$$

It will be seen that $\dim \text{convex hull}(\mathcal{U}_A(x))$ is a better measure of the size if the set of generators throw x than just $N_A(x)$.

6.11. **Proposition.** *For $k \geq 1$ we have*

$$\{x \in \mathrm{Cut}(A) : N_A(x) \geq k\} \subseteq \mathrm{Cut}^{[k]}(A).$$

For small values of k this can be improved:

$$\{x \in \mathrm{Cut}(A) : N_A(x) \geq k\} = \mathrm{Cut}^{[k]}(A) \qquad \textit{for} \qquad 1 \leq k \leq 3.$$

Exercise 8. Prove this. □

The set $\mathrm{Cut}^{[2]}(A)$ of points that are on two or more A-generators is often called the ***strict cut locus***. Note that the result of Beem-Królak Theorem 6.8 implies that the $\mathrm{Cut}^{[2]}(A)$ is exactly the set of points $x \in \mathbf{R}^n \setminus A$ where ρ_A is not differentiable.

6.12. **Proposition.** *The strict cut locus $\mathrm{Cut}^{[2]}(A)$ of A is dense in the cut locus $\mathrm{Cut}(A)$. Therefore for any open set U we have that $\mathrm{Cut}(A) \cap U \neq \varnothing$ if and only if $U \cap \{x : d\rho_A(x)$ does not exist$\} \neq \varnothing$.*

Proof. If $\mathrm{Cut}^{[2]}(A)$ is not dense in $\mathrm{Cut}(A)$ then there is an open set U, disjoint from A so that $U \cap \mathrm{Cut}(A) \neq \varnothing$, but $U \cap \mathrm{Cut}^{[2]}(A) = \varnothing$. But then for all $x \in U$ we have $N_(x) = 1$ and therefore ρ_A is differentiable on U by Theorem 6.8. But by Proposition 6.10 this implies U is disjoint from the cut locus, contradicting $U \cap \mathrm{Cut}(A) \neq \varnothing$. □

This can be combined with Theorem 1 to give the following:

6.13. **Theorem.** *Let*

$$\mathcal{N} := \big\{A \in \mathcal{K} : \{x : d\rho_A(x) \text{ does not exist}\} \text{ is dense in } \mathbf{R}^n \setminus A\big\}.$$

Then, with the notation of Theorem 1, $\mathcal{N} = \mathcal{D}$ and therefore $\mathcal{N}$ is a dense G_δ in $\mathcal{K}$.

Exercise 9. Prove this as a corollary to Proposition 6.12. □

We can now refine Proposition 6.9 by increasing the regularity to $C^{1,1}_{\mathrm{Loc}}$.

6.14. **Theorem.** *Let $A \in \mathcal{K}$ and let U be an open set disjoint from A so that ρ_A is differentiable on U (which, by Proposition 6.12 is equivalent to $\mathrm{Cut}(A) \cap U = \varnothing$). Then ρ_A is locally a $C^{1,1}$ function in U. (That is locally the gradient is a Lipschitz function).*

Proof. Let $a_1 \in A$ and define $\varphi_1(x) = \|x - a_1\|$. Then as in the proof of Theorem 6.4 we have

$$\rho_A(x) \leq \|x - a_1\| = \rho_1(x)$$

for all $x \in \mathbf{R}^n$. Let $x_0 \in \mathbf{R}^n$ and let $\rho_0(x) = \rho_A(x_0) - \|x - x_0\|$. The the inequality (2.2) implies $\rho_A(x_0) \leq \rho_A(x) + \|x - x_0\|$ and therefore

$$\varphi_0(x) = \rho_A(x_0) - \|x - x_0\| \leq \rho_A(x).$$

We now make special chooses of a_1 and x_0. Let $c\colon [0,b] \to \mathbf{R}^n$ be an A-minimizing segment. Let $a_1 = c(0) \in A$ and $x_0 = c(b)$. Then for $t \in (0,b)$ we have

$$\rho_A(c(t)) = t = \|c(t) - c(0)\| = \|c(t) - a_1\| = \varphi_1(c(t)),$$

and

$$\begin{aligned} \rho_a(c(t)) = t = b - (b-t) &= \rho_A(c(b)) - \|c(b) - c(t)\| \\ &= \rho_A(x_0) - \|c(t) - x_0\| = \varphi_0(c(t)). \end{aligned}$$

Thus for this choice of a_1 and x_0 the function φ_0 is a lower support function and φ_1 is an upper support function to ρ_A at $c(t)$ whenever $0 < t < b$. Let B be an open ball in U with $\operatorname{dist}(B, \partial U) = r > 0$. Then for any $x \in B$ there will be an A-minimizing segment $c\colon [0,b] \to \mathbf{R}^n$ so that $x = c(t)$ for some t and $0 < t, b-t < r$ (for the A-generator through x has no end point in U and therefore $\operatorname{dist}(B, \partial U)$ implies we can extend to at least a distance r past any point $x = c(t) \in B$). Let $a_1 = c(0)$ and $x_0 = c(b)$ and let φ_0 be the support functions ρ_A at $x = c(t)$ as above. By (6.1) there are bounds on the Hessian $D^2\varphi_0$ and $D^2\varphi_1$ hold on all of B (and in fact only depend on r). By [6, Prop 1.1 p. 7] this implies that $\rho_A\big|_B$ is $C^{1,1}$. As U is the union of open balls that have positive distance from the boundary this shows that the restriction of ρ_A to U is $C^{1,1}$. $\square$

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