A RIGIDITY THEOREM FOR CONVEX SURFACES

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I learned of the following very pretty result and its proof from Calabi [2].

Theorem 1. *Let (S^2, g) be the two dimensional sphere with a metric of class $C^{1,1}$ whose Gaussian curvature satisfies $0 \leq K \leq 1$. Then any simple closed geodesic γ on (S^2, g) has length at least 2π. If the length of γ is 2π, then (S^2, g) is isometric to the standard round sphere (S^2, g_0) and γ is a great circle on (S^2, g_0) or (S^2, g) is isometric to a circular cylinder of circumference 2π capped by two unit hemispheres and γ is a belt around the cylinder (see Figure 1). Thus if K is continuous (for example when g is at least C^2) or if $K > 0$ then (S^2, g) is isometric to the standard round sphere.*

Remark 1. The lower bound on the length of a simple closed closed geodesic is well known (see Remark 3), it is the rigidity result that is of interest here.

It is interesting that requiring the metric to have regularity class C^2 gives a different class of extremal surfaces than the $C^{1,1}$ case. Likewise if $K > 0$ the only extremal surface is the standard sphere.

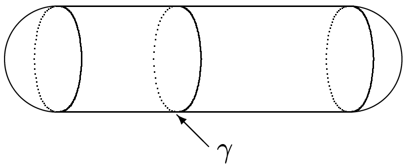

FIGURE 1

Lemma 1. *Let $k(t)$ be an L^∞ function on $[0, \infty)$ so that $0 \leq k(t) \leq 1$ and let $y(t)$ be defined by the initial value problem*

$$y''(t) + k(t)y(t) = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Denote the smallest positive zero of y by β (it may be that $\beta = \infty$). Then $0 \leq -y'(t) \leq 1$ for $0 \leq t \leq \beta$. If $y'(t_0) = -1$ for some $t_0 \in [0, \beta]$, then $t_0 = \beta < \infty$, $\beta \geq \pi/2$ and

$$y(t) = \begin{cases} 1, & 0 \leq t < \beta - \pi/2; \\ \cos(t - (\beta - \pi/2)), & \beta - \pi/2 < t \leq \beta. \end{cases}$$

$$k(t) = \begin{cases} 0, & 0 \leq t < \beta - \pi/2; \\ 1, & \beta - \pi/2 < t \leq \beta. \end{cases}$$

If k is continuous, for example if y is C^2, and $y'(t_0) = -1$ with $t_0 \leq \beta$, then $t_0 = \beta = \pi/2$, $k \equiv 1$ and $y(t) = \cos(t)$ on $[0, \pi/2]$.

Proof. First note on the interval $[0,\beta)$

$$(y')' = y'' = -ky \le 0$$

as $y > 0$ and $k \ge 0$ on $[0,\beta)$. Thus y' is monotone decreasing on $[0,\beta]$. As $y'(0) = 0$ this implies $y' \le 0$ on $[0,\beta]$. Thus

$$\big(y^2 + (y')^2\big)' = 2yy' + 2y'y'' = 2yy' - 2y'ky = 2yy'(1-k) \le 0 \tag{1}$$

on $[0,\beta)$ as $k \le 1$ and $y' \le 0$. Using the initial conditions for y and continuity the last inequality implies

$$y^2 + (y')^2 \le 1 \quad \text{on} \quad [0,\beta]. \tag{2}$$

These inequalities imply $0 \le -y' \le 1$ on $[0,\beta]$.

If $t_0 \in [0,\beta]$ and $y'(t_0) = -1$, then the inequality (2) implies $y(t_0) = 0$. But from the definition of β as the smallest positive zero of y this implies $t_0 = \beta$. Then $y(\beta)^2 + y'(\beta)^2 = 1$ and equation (1) yields $y'(1-k) \equiv 0$ on $[0,\beta)$. As y' is monotone decreasing on $[0,\beta)$, there is a point $t_1 \in [0,\beta)$ so that $y' \equiv 0$ on $[0,t_1]$ and $0 > y' > -1$ on (t_1,β). Then on $[0,t_1]$ we have $y \equiv 1$ and $k \equiv 0$ (as $ky = -y'' = 0$). Also $y'(1-k) \equiv$ and $y' \ne 0$ on (t_1,β) implies $k \equiv 1$ on (t_1,β). But $y(t_1) = 1$ and $y'(t_1) = 0$ so $y(t) = \cos(t - t_1)$ on (t_1,β). As $y(\beta) = 0$ this implies $t_1 = \beta - \pi/2$ on (t_1,β). This completes the proof. $\square$

Proof of the theorem. Let $c : [0,L] \to S^2$ be a unit speed parameterization of the closed geodesic γ, and let $\mathbf{n}$ be a unit normal along c. For each $s \in [0,L]$ let $\beta(s)$ be the cut distance from the curve γ along the geodesic $t \mapsto \exp_{c(s)}(t\mathbf{n}(s))$. Define map $F(s,t)$ on the set of ordered pairs (s,t) with $s \in [0,L]$ and $0 \le t \le \beta(s)$ by

$$F(s,t) = \exp_{c(s)}(t\mathbf{n}(s)), \qquad 0 \le s \le L, \quad 0 \le t \le \beta.$$

Then s,t are Fermi coordinates on the disk M bounded by γ and with inner normal $\mathbf{n}$. In these coordinates the metric g, Gaussian curvature K and the area form dA are given by

$$g = E^2\,ds^2 + dt^2, \quad K = \frac{E_{tt}}{E}, \quad dA = E\,ds\,dt.$$

And because c is a geodesic $E(s,0) \equiv 1$ and $E_t(s,0) \equiv 0$. Thus for fixed s the function $y(t) := E(s,t)$ satisfies $y'' + Ky = 0$, $y(0) = 1$, and $y'(0) = 0$ as in the lemma.

Now apply the Gauss-Bonnet theorem to the disk M. As the boundary is a geodesic, the boundary term of the formula drops out:

$$\begin{aligned} 2\pi &= \int_M K\,dA \\ &= \int_0^L \int_0^{\beta(s)} \frac{E_{tt}}{E} E\,dtds \\ &= \int_0^L \int_0^{\beta(s)} E_{tt}\,dtds \\ &= \int_0^L (-E_t(s,\beta(s)))\,ds \qquad &&(\text{as } E_t(s,0)=0) \\ &\leq \int_0^L 1\,ds &&(\text{by the lemma}) \\ &= L. \end{aligned}$$

which proves the required lower bound on the length of γ. If $L = 2\pi$, then $E_t(s,\beta(s)) = -1$ for all $s \in [0,L]$. Again by the lemma in the coordinates s,t on M

$$K(s,t) = \begin{cases} 0, & 0 \leq t < \beta(s) - \pi/2; \\ 1, & \beta(s) - \pi/2 < t < \beta(s). \end{cases}$$

Let $M_{+1} = \{x \in M : K(x) = +1\} = \{\exp_{c(s)}(t\mathbf{n}(s)) : s \in [0,2\pi], \beta(s) - \pi/2 < t \leq \beta(s)\}$. Let $s_0 \in [0,2\pi]$ be a point where $\beta(s)$ is maximal. Then the open disk $B(x_0,\pi/2)$ of radius $\pi/2$ about $x_0 := \exp_{c(s_0)} \beta(s_0)\mathbf{n}(s_0)$ is contained in M_{+1}, for if not it would meet ∂M_{+1} at some point $\exp_{c(s)}((\beta(s) - \pi/2)\mathbf{n}(s))$ and this point is a distance of $\beta(s) - \pi/2$ from γ. Thus the distance of $x_0 = \exp_{c(s_0)}(\beta(s_0)\mathbf{n}(s_0))$ to γ is less than $\pi/2 + (\beta(s) - \pi/2) = \beta(s)$, which contradicts the maximality of $\beta(s_0)$. Thus $B(x_0,\pi/2) \subseteq M_{+1}$. But using the Gauss-Bonnet theorem and $K \equiv +1$ on M_{+1}

$$2\pi \geq \int_{M_{+1}} K\,dA = \operatorname{Area}(M_{+1}) \geq \operatorname{Area}(B(x_0,\pi/2)) = 2\pi.$$

So $M_{+1} = B(x_0,\pi/2)$. From this it is not hard to show $s \mapsto \beta(s)$ is constant and thus the disk M bounded by γ and with inner normal $\mathbf{n}$ is a cylinder of circumference 2π capped at one end with a hemisphere (see Figure 2).

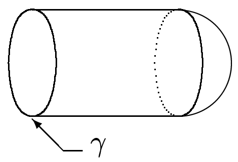

FIGURE 2

The same argument applied to the disk bounded by γ and having $-\mathbf{n}$ as inward normal shows (S^2,g) is two of these capped cylinders glued together along γ, which is equivalent to the statement of the theorem. $\square$

Corollary 1. *Let $(\mathbf{RP}^2, g)$ be the real projective space with metric g so the Gauss curvature K of g satisfies $0 \leq K \leq 1$. Then any curve α in $\mathbf{RP}^2$ that is not homotopic to 0 has length at least π. If α has length π then it* *is a simple closed geodesic and* $(\mathbf{RP}^2, g)$ *is isometric to the standard metric on* $\mathbf{RP}^2$*, or* $(\mathbf{RP}^2, g)$ *is isometric to the capped cylinder on Figure 2 with antipodal points on the boundary curve identified. In this case* α *is the image of the boundary curve.*

Proof. By standard methods form the calculus of variations the non-trivial free homotopy class of $(\mathbf{RP}^2, g)$ has a representative α of minimal length and α is a simple closed geodesic. Let $p : S^2 \to \mathbf{RP}^2$ be the covering map and let $\gamma = p^{-1}[\alpha]$. If S^2 is given the pull back metric p^*g, then the theorem applies to γ. Thus the length of γ is at least 2π. But γ double covers α so the length of α is at least π. If the length of α is π then the length of γ is 2π. Thus the equality case of the theorem holds. It is not hard to translate the rigidity statement of Theorem 1 into the one here. □

We note Theorem 1 is "dual" to a result of Toponogov [6].

Theorem 2 (Toponogov, 1959)**.** *Let* (S^2, g) *be the two dimensional sphere with a metric so that the Gaussian curvature satisfies* $1 \leq K$*. Then any simple closed geodesic* γ *has length at most* 2π*, and if the length of* γ *is* 2π *then* (S^2, g) *is isometric to the standard sphere* (S^2, g_0) *and* γ *is a great circle.*

Corollary 2. *Let* $(\mathbf{RP}^2, g)$ *be the real projective plane with a metric that satisfies* $K \geq 0$*. Let* α *be the shortest curve not homotopic to 0. Then* α *is a simple closed geodesic and the length of* α *is at most* π*. If the length of* α *is* π *then* $(\mathbf{RP}^2, g)$ *is isometric to the standard metric on* $\mathbf{RP}^2$*.*

Remark 2. Let (S^2, g) have Gaussian curvature satisfying $0 < a \leq K \leq b$. Then Theorems 1 and 2 show any simple closed geodesic γ satisfies

$$\frac{2\pi}{\sqrt{b}} \leq \mathrm{Length}(\gamma) \leq \frac{2\pi}{\sqrt{a}}.$$

Remark 3. If one is only interested in the length of closed geodesics, there are higher dimensional versions of Theorem 1. If (M, g) is a compact orientable Riemannian manifold of even dimension with sectional curvatures satisfying, $0 < K_M \leq 1$, then Klingenberg has shown every closed geodesic has length $\geq 2\pi$. This is equivalent to his well known lower bound on the injectivity radius of compact oriented even dimensional manifolds. For a proof see the book [3, Chapter 5]. For odd dimensional manifolds anther theorem of Klingenberg's implies if (M, g) is a compact simply connected manifold of whose sectional curvature satisfies $1/4 < K_M \leq 1$, then any closed geodesic of (M, g) has length at least 2π. Again a proof can be found in [3, Chapter 5]. (The original proofs of Klingenberg are in [4, 5].) We also note that in dimension 3 for any $\varepsilon > 0$ there are examples of metrics g on $M = S^3$ (due to Berger) so that the sectional curvatures satisfy $1/9 - \varepsilon \leq K_M \leq 1$, but $(S^3, g) = (M, g)$ has a geodesic of length less than 2π. See [3, Example 3.35, page70]. To the best of my knowledge there is no known rigidity result in the above theorems.

There is a higher dimensional version of the rigidity part of Theorem 1. In [1] it is shown:

Theorem 3 (Andersson and Howard)**.** *Let (M, g) be a complete Riemannian manifold of dimension at least three and so the sectional curvatures of (M, g) satisfy $K_M \leq 1$. Let U be any open connected neighborhood of the equator S^{n-1} in the standard sphere (S^n, g_0). Then any local isometry $\phi : (U, g_0) \to (M, g)$ extends to a local isometry $\widehat{\phi} : (S^n, g_0) \to (M, g)$. Thus if such a local isometry $\phi : (U, g_0) \to (M, g)$ exists, then the sectional curvature of (M, g) is identically one.*

Here the curvature assumption is weakened ($K \leq 1$ instead of $0 \leq K \leq 1$) but the existence of a simple closed geodesic of length 2π is replaced by the much stronger condition of the existence of a local isometry $\phi : (U, g_0) \to (M, g)$ for some neighborhood U of an equator $S^{n-1} \subset S^n$. It is likely this condition can be weakened. The proof of theorem 3 is an induction on the dimension and the base case was a weak version of Theorem 1.

References

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