HYPERSURFACES WITH PRESCRIBED MEAN CURVATURE.

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Contents

1. Introduction

This contains the beginnings of an existence theory for viscosity type solutions for prescribed mean curvature to boundaries of domains on Riemannian manifolds. These solutions can have geometric singularities of types that occur naturally in geometric (for example variational) problems. There is an outline, close to a complete proof, of a basic existence theorem, and just a hint of of what should happen with regularity.

2. The k-Mean Curvature in the Viscosity Sense.

2.1. **The k-mean curvature for C^2 boundaries.** Let M be a complete n dimensional Riemannian manifold. For any domain D in M with C^2 boundary ∂D let $\mathbf{n}$ be the outward pointing unit normal to ∂D. Letting ∇ be the connection (or covariant derivation) of the metric on M we define the ***second fundamental form*** $\mathit{I\!I}$ of ∂D by

$$\mathit{I\!I}(X,Y) := \langle \nabla_X Y, \mathbf{n} \rangle$$

where X, Y are C^1 vector fields tangent to ∂D. As usual this is a symmetric $(0,2)$ tensor. The ***Weingarten map***, or ***shape operator***, A of ∂D is the field of linear maps on $T(\partial D)$ given by

$$AX := -\nabla_X \mathbf{n}.$$

This is related to $I\!I$ by

$$I\!I(X,Y) = \langle AX, Y\rangle.$$

The ***mean curvature*** H of ∂D is then

$$H := \frac{1}{n-1}\operatorname{trace} A.$$

In terms of $I\!I$ this is given by

$$H = \frac{1}{n-1}\sum_{i=1}^{n-1} I\!I(e_i, e_i)$$

where $e_1, \dots, e_{n-1}$ is a locally defined orthonormal moving frame on ∂D. If $B^n(r)$ is the open ball of radius r in $M = \mathbf{R}^n$ then with the conventions here $H = -1/r$. (The negative is because we are using the outward normal and $\partial B^n(r)$ curves toward the interior of $B^n(r)$.)

Let $\mathbb{U}(M)$ be the sphere bundle of unit vectors tangent to M.

2.1. **Definition.** Let $k \colon \mathbb{U}(M) \to \mathbf{R}$ be a C^1 function. Then for any domain $D \subset M$ with C^2 boundary the ***k-mean curvature*** of ∂D is the function on ∂D given by

$$\theta^{\partial D}(p) := H(p) + k(\mathbf{n}(p))$$

where H is the mean curvature of ∂D. (If the domain ∂D is clear from context then this will be written as simply θ.) □

It is also useful to have a notation for the k-mean curvature computed with respect to the inward normal rather than the outward normal. The mean curvature H changes sign when $\mathbf{n}$ is replaced by $-\mathbf{n}$. So the k-mean curvature with respect to the inward normal is

$$\theta_-^{\partial D}(p) = -H(p) + k(-\mathbf{n}(p)).$$

Again when the domain is clear from context this will be written as simply θ_-.

2.2. **The definition of k-mean curvature in the viscosity sense.** We now wish to extend this definition to general domains in such a way that it agrees with this classical definition on domains with C^2 boundary, but still allows for naturally occurring geometric singularities. The definition we give a geometric version viscosity solutions (cf. [2, 1]). We will be replacing functions by boundaries of open sets, and upper and lower support functions by inner and outer support domains.

2.2. **Definition.** Let D be an open set $p \in D$. Then U is an ***inner support domain*** (respectively an ***outer support domain***) iff U is an open set with $U \subset D$ (respectively $U \cap D = \varnothing$), the boundary ∂U is a C^2 hypersurface in a neighborhood of p, and $p \in \partial U$. □

We will sometimes say that U ***touches*** ∂D ***from the inside*** (respectively ***touches*** ∂D ***from the outside***).

2.3. **Definition.** Let D be an open set in M. Then we say that $K \leq 0$ ***on*** ∂D ***in the viscosity sense*** (respectively $K \geq 0$ ***on*** ∂D ***in the viscosity sense***) for any point $p \in \partial D$ and any inner support (respectively outer) domain U to ∂D at p the inequality $\theta^{\partial D}(p) \leq 0$ (respectively $\theta_{-}^{\partial D}(p) \geq 0$) holds. If both $\theta^{\partial D} \geq 0$ and $\theta^{\partial D} \leq 0$ in the viscosity sense, then $\theta^{\partial D} = 0$ ***in the viscosity sense***.

In the definition of $\theta \geq 0$ in the viscosity sense we use $\theta_{-}^{\partial D}(p) \geq 0$, that the k-mean curvature is computed with respect to the inward normal to the outer support function U. This is because when ∂D is smooth the inward normal to an outer support function is the outward normal to ∂D at a point of contact.

There may be many points $p \in \partial D$ that have no inner support domains at p. For example if D in the interior of a triangle in the plane then there are no inner support domains at any of the vertices of the triangle. Therefore $\theta \leq 0$ in the viscosity sense is weak notion of the inequality $K \leq 0$ as in only gives direct information at points where there is an inner support domain.

This can be put picturesquely as follows. A domain D fails to have $\theta \leq 0$ in the viscosity sense if and only if a blindfolded mathematician with C^2 fingertips can "feel" it does not have $\theta \leq 0$ by finding a point where he/she can tell $\theta > 0$ by touch. At points not accessible to C^2 fingertips the mathematician will not be able get any information. Likewise a domain D fails have $\theta \geq 0$ in the viscosity sense if and only if our blindfolded mathematician can feel that $\theta < 0$ at some point.

As a first example we have the elementary result whose proof is left to the reader.

2.4. **Proposition.** *Let D be a domain in M with C^2 boundary. Then $K \leq 0$ (respectively $K \geq 0$ or $K = 0$) in the viscosity sense if and only if $K \leq 0$ (respectively $K \geq 0$ or $K = 0$) in the classical sense.* □

As an example let D be the interior of convex polyhedron in $\mathbf{R}^n$. Then the standard mean curvature H satisfies $H \leq 0$ on ∂D as on the $(n-1)$-dimensional faces of ∂D we $H = 0$ in the classical sense. The lower dimensional faces can not be touched from within D by C^2 inner support domains. This accounts for all the points of ∂D and so $H \leq 0$ in the viscosity sense. (More generally it is easily checked that any convex open set has mean curvature with $H \leq 0$ in the viscosity sense.)

A more interesting example is $D = \{(x, y) : xy > 0\}$ (that is the union of the first and third quadrants). Let $\theta = H$ so that θ is just the curvature in the usual sense. Then $\theta = 0$ on ∂D as at all points other than the origin $\theta = 0$ in the classical sense. At the origin there are no inner or out support domains so $\theta = 0$ holds on all of ∂D. This extends to higher dimensions. Let $D := \{(x^1, \dots, x^n) \in \mathbf{R}^n : x^1 x^2 > 0\}$. Then ∂D has mean curvature $= 0$ in the viscosity sense, even thought there is a singularity along the codimension two subspace $\{x^1 = x^2 = 0\}$ (codimension one in ∂D).

This notation is also related to variational solutions to least area type problems.

2.5. **Proposition.** •*2.1* *Let M be a complete n-dimensional Riemannian manifold and $D \subset M$ an open set so that its boundary satisfies $\mathcal{H}^{n-1}(\partial D) \leq \mathcal{H}^{n-1}(\partial D')$ for any other open with $\mathrm{Vol}(D') = \mathrm{Vol}(D)$. Then there is a constant H_0 so that $H - H_0 = 0$ in the viscosity sense. (Here $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.)*

2.1: Look in Federer and get the definitions right (i.e. use currents and the like). Also fill in detail of proof and check if stationary (as opposed to minimizing) is enough.

Proof. If is a standard result ∂D will be a smooth hypersurface with constant mean curvature H_0 except for a singular set $\partial D_{\mathrm{sing}}$ which will have Hausdorff codimension at least seven in ∂D. Let $p \in \partial D$ and let U be an inner support domain to ∂D at p. Then as ∂U is C^2 in a neighborhood of p its tangent cone at p will be the hyperplane $T(\partial U)_p \subset T(M)_p$ and the tangent cone to ∂D at p will be a cone over a minimal verity in the unit sphere of $T(M)_p$. As ∂D is disjoint from U the tangent cone to ∂D will lie in one of the two closet half planes of $T(M)_p$ determined by $T(\partial U)$. But a minimal cone contained in a half space is a hyperplane. Thus the tangent cone to ∂D at p is $T(U)_p$. By the regularity theory this implies that p is a smooth point of ∂D, that is $p \notin \partial D_{\mathrm{sing}}$. As $H \equiv H_0$ on $\partial D \smallsetminus \partial D_{\mathrm{sing}}$ in the classical sense we are done. $\square$

2.3. **Elementary properties.** One use of mean curvature is to estimate focal distances and this give upper bounds on distances. The following shows that the viscosity version of mean curvature is good enough for this.

2.6. **Proposition.** *Let M be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$. Let $D \subset M$ be an open set with mean curvature $H \geq 1/r_0$ in the viscosity sense. Then any point of M is at a distance $\leq r_0$ from $\overline{D}$.*

Proof. Toward a contradiction assume that there is a point $q \in M$ with $\mathrm{dist}(q, \overline{D}) > r_0$. Let $r_1 = \mathrm{dist}(q, \overline{D})$ and let $p \in \partial D$ be a point with $\mathrm{dist}(q,p) = r_1$. Let $\gamma \colon [0, r_1] \to M$ be a unit speed geodesic with $\gamma(0) = p \in \partial D$ and $\gamma(r_1) = q$. Let r_2 be so that $r_0 < r_2 < r_1$. The geodesic γ is minimizing and so $\gamma(r_2)$ has no cut point on the segment $\gamma\big|_{[0,r_2]}$ which implies that the open geodesic ball $B(\gamma(r_2), r_2)$ is smooth near $\gamma(0) = p$. Also by the triangle inequality $B(\gamma(r_2), r_2) \subseteq B(\gamma(r_1), r_1)$ and $B(\gamma(r_1), r_1) \cap D = \varnothing$ as $r_1 = \mathrm{dist}(q, \overline{D})$. Therefore $B(\gamma(r_2), r_2)$ is an outer support domain for ∂D at p. By standard comparison results the mean curvature of $\partial B(\gamma(r_2), r_2)$ with respect to the inward normal satisfies $1/r_2 \geq H^{\partial B(\gamma(r_2),r_2)}_{\mathrm{inward}}$. But the definition of $H^{\partial D} \geq 1/r_0$ in the viscosity sense requires $H^{\partial B(\gamma(r_2),r_2)}_{\mathrm{inward}} \geq 1/r_0$.

These inequalities implies $1/r_2 \geq r_0$ which is impossible as $r_0 < r_2$. This contradiction completes the proof. □

Of more interest is the Lorentzian version of this.

2.7. **Proposition.** *The Penrose singularity theorem holds with the trapped region having its divergence defined in the viscosity sense rather then in the classical sense.*

Proof. Come back to this. It should be an easy variant on the usual proof, which in turn is just a variation on the last result. □

3. Trapped Regions and The Main Existence Theorem

3.1. **Constructing viscosity solutions to $\theta \geq$ and $\theta \leq 0$.** Our goal is to construct domains D with $\theta^{\partial D} = 0$ by a geometric variant of Perron's method for elliptic equations. Instead of taking a supremum of subsolutions, we will take unions of domains with $\theta > 0$. We give such domains a name (the terminology comes from the trapped surfaces of general relativity).

3.1. **Definition.** A domain $D \subset M$ is a ***trapped domain*** iff $\theta^{\partial D} > 0$ on all of ∂D and $\overline{D}$ is compact. (Note we do not assume that either D or ∂D is connected.) □

The following is the analogue of the supremum of subsolution being subsolution.

3.2. **Proposition.** *Let M be a complete Riemannian manifold and let $\{D_\alpha : \alpha \in A\}$ be a collection of trapped regions in M. Then the union $D := \bigcup_{\alpha\in A} D_\alpha$ has $\theta \geq 0$ in the viscosity sense. (But D need not have compact closure.)*

Proof. Toward a contradiction, assume that ∂D does not have $\theta \geq 0$ in the viscosity sense. There there is a point $p \in \partial D$ and a outer support region V to ∂D at p so that ∂V is C^2 in a neighborhood of p and $\theta_-^{\partial V}(p) < -4\varepsilon$ for some small $\varepsilon > 0$. Then by continuity we can assume that $\theta^{\partial V} < -3\varepsilon$ in some neighborhood of p. Then it is possible to choose an open subset W of V so that W is also an outer support region for ∂D at p, so that $\partial W \cap \partial V = \{p\}$, and also $\theta_-^{\partial W} < -2\varepsilon$ near p. (See Figure 1.) Let W_r be

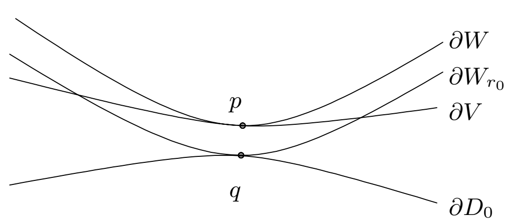


Figure 1

the set of point on M that have Riemannian distance $< r$ from $\overline{W}$. As $\overline{W}$

only meets ∂V at p, for sufficiently small r all of W_r will be a subset of V except near p (again see Figure 1). Also as W is C^2 near p (this is part of the definition of W being an outer support region to ∂D at p) and so for sufficiently small r we have that near p the boundary ∂W_r is a parallel hypersurface to ∂W and therefore also C^2. So choose r_1 so small that if $r < r_1$ all of ∂W_r which is not contained in V is C^2 and has $\theta^{\partial W_r} < \varepsilon$ (this is possible by continuity as $\partial W_r \to \partial W$ in the C^2 topology as $r \searrow 0$). Now as W_{r_1} is an open neighborhood of p by the definition of D there is a trapped domain $D_0 \in \{D_\alpha : \alpha \in A\}$ so that $\overline{W_{r_1}} \cap D_0 \neq \varnothing$. Now let q be a point of $\overline{D}_0$ that is a minimal distance from $\overline{W}$ and let r_0 be the distance of q from $\overline{W}$ (which is the same as the distance from D_0 to $\overline{W}$). Then ∂D_0 and ∂W_{r_0} are tangent at q. But this contradicts the strong maximal principle as $\theta > 0$ on ∂D_0 and $\theta_- < -\varepsilon < 0$ at q on ∂W_{r_0} (recall that we are computing θ_- with respect to the inward normal at on ∂W_{r_0}). This completes the proof. □

We need a method of constructing large trapped regions in M from smaller ones. The following, which is implicit in the paper [4] of Kriele-Hayward, does this.

3.3. **Proposition** (Kriele-Hayward [4]). *Let $D_1, \dots, D_N \subset M$ be a finite number of open sets with C^2 boundaries. Assume that $\theta^{\partial D_k} > 0$ on $\partial D_k \setminus \bigcup_{j\neq k} D_j$. Then there is a region D with C^2 boundary, $\theta^{\partial D} > 0$ and with $D \supset D_1 \cup \cdots \cup D_N$.*

3.4. *Remark.* As the proof will make clear it is not necessary to assume that all of ∂D_k is C^2, but only that $\partial D_k \setminus \bigcup_{j\neq k} D_j$ is contained in a part of the boundary that is C^2. The result will be applied in this more general form below. □

Proof. By use of induction it is enough to assume that $N = 2$ and that $\theta^{\partial D_1} > 0$ on $\partial D_1 \setminus D_2$ and $\theta^{\partial D_2} > 0$ on $\partial D_2 \setminus D_1$.

We would like to assume that ∂D_1 and ∂D_2 intersect transversally. If they do not than as the condition $\theta > 0$ is open in the C^2 topology and the space of embeddings of ∂D_2 which are transverse to ∂D_1 is open and dense in the space of embeddings into M, we can replace D_2 be a slightly larger domain D_2^* with so that $\theta > 0$ on $\partial D_2^* \setminus D_1$ and so that ∂D_1 and ∂D_2^* meet transversely. Renaming D_2^* as D_2 we can assume that ∂D_1 and ∂D_2 meet transversely. (See Figure 2.) The idea is to span the corner along $\partial D_1 \cap \partial D_2$ be a smooth hypersurface as in Figure 3. Now that the spanning hypersurface curves away from $D_1 \cup D_2$ and therefore with some care we can arrange for $\theta > 0$ on this strip.•3.1 (See the proof of [4, Lemma 6 p. 1599].) □

3.1: It would be worth while to find a short proof other than the one in [4]. Besides I am not sure I trust them to get it right.

Taking infinite unions does not lead to quit such a clean result.

3.5. **Proposition.** *Let M be a complete Riemannian manifold and $\{D_\alpha : \alpha \in A\}$ be a collection of trapped regions in M. Then either the union*

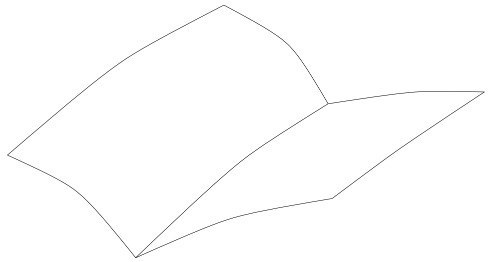


FIGURE 2. A part of $\partial(D_1 \cup D_2)$ near a part of $\partial D_1 \cap \partial D_2$. The domain $D_1 \cup D_2$ is below the pictured surface.

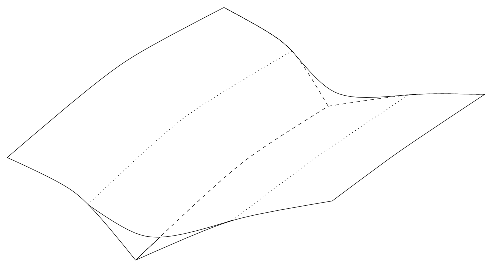


FIGURE 3. Smoothing the corner of $\partial(D_1 \cup D_2)$ along $\partial D_1 \cap \partial D_2$.

$D := \bigcup_{\alpha \in A} D_\alpha$ *has* $\theta \leq 0$ *in the viscosity sense, or there is a trapped region* D^* *that contains a point not in* D.

Proof. If ∂D does not have $\theta^{\partial D} \leq 0$ in the viscosity sense then there is a point $p \in \partial D$ and an inner support domain V to ∂D at p so that $\theta^{\partial V}(p) > 4\varepsilon$ for some small positive number p. By continuity we can assume that $\theta > 3\varepsilon$ on some neighborhood of p in ∂V. Then there is an open subset W of V so that W is also an outer support region for ∂D at p, so that $\partial W \cap \partial V = \{p\}$, and also $\theta > 2e$ near p. As in the proof of Lemma 3.2 if W_r is the set of points of M with distance $< r$ from $\overline{W}$ for r sufficiently small ∂W_r will be a C^2 hypersurface near p and by taking r still smaller we can assume that $\theta > \varepsilon$ on all of $\partial W_r \setminus V$. (See Figure 4) Without loss of generality

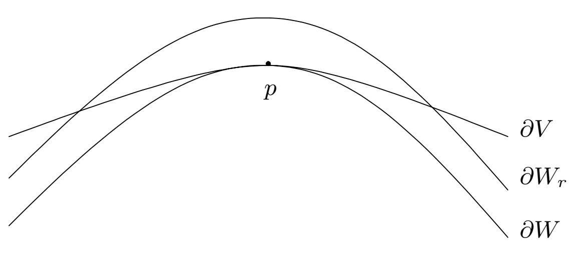


FIGURE 4

we can assume that W, and therefore W_r, has compact closure. (If not interest W with a large geodesic ball centered at p.) Let $\iota_\rho V := \{q \in V :$

$\operatorname{dist}(q, \partial V) > \rho\}$. By choosing ρ small enough we can assume that $\partial W_r \setminus \iota_\rho$ is a C^2 hypersurface with $\theta > 0$. As W_r has compact closure the same is true of $W_r \cap \iota_\rho V$. Each point of $\overline{W_r \cap \iota_\rho V}$ is contained in some trapped domain and trapped domains are open so there is a finite set of trapped domains $D_1, \dots, D_N \subseteq \{D_\alpha : \alpha \in A\}$ so that $\overline{W_r \cap \iota_\rho V} \subset D_1 \cup \cdots \cup D_N$. Proposition 3.3 implies there is a trapped domain $\tilde{D}$ with $\overline{W_r \cap \iota_\rho V} \subset \tilde{D}$. But then $\partial W_r \setminus \tilde{D} \subseteq \partial W_r \setminus \iota_\rho$ and so $W_r \setminus \tilde{D}$ is a C^2 hypersurface with $\theta > 0$. Therefore we can apply Proposition 3.3 again and conclude $\tilde{D} \cup W_r$ is contained is some trapped domain D^*. But then D^* contains $p \notin D$. This completes the proof. □

3.2. **The extremal trapped region.** We now wish to construct a domain $\theta = 0$ in the viscosity sense and which also is largest. The maximality property of this solution should force it to have better regularity properties than most solutions.

3.6**. Definition.** The ***extremal trapped region*** of M is $\mathcal{D} = \mathcal{D}^\theta$ given by

$$\mathcal{D} := \bigcup \{D \subset M : D \text{ is a trapped region of } M\}.$$

That is $\mathcal{D}$ is the set of points of M are in the interior of at least one trapped domain. □

3.7**. Theorem** (Main Existence Theorem)**.** *The extremal trapped region satisfies $\theta^{\partial \mathcal{D}} = 0$ in the viscosity sense.*

Proof. That $\theta^{\partial \mathcal{D}} \geq 0$ in the viscosity sense follows from Proposition 3.2. If we do not have $\theta^{\partial \mathcal{D}} \leq 0$ in the viscosity sense then by Proposition 3.5 there is a trapped domain D^* that contains a point not in $\mathcal{D}$. But the definition of $\mathcal{D}$ implies that $D^* \subseteq \mathcal{D}$ and so this is impossible. Therefore $\theta^{\partial \mathcal{D}} \leq 0$ in the viscosity sense. This completes the proof. □

We now give conditions that imply the extremal trapped region is bounded (that is it has compact closure).

3.8**. Definition.** The Riemannian manifold M has a ***barrier*** for θ iff there is an open set U of M and a C^2 function $f\colon \overline{U} \to \mathbf{R}$ so that

(1) There is a constant C_0 so that $f = C_0$ on ∂U and $f > C_0$ on U,
(2) f has no critical points in U, and
(3) the sets $f^{-1}\big[(-\infty, r)\big]$ have $\theta \leq 0$ on their boundaries for $r > C_0$. □

3.9**. Proposition.** *Let M have a barrier for θ, then every trapped region D is a subset of $M \setminus U$. Therefore if $M \setminus U$ is compact then $\mathcal{D}$ has compact closure.*

Proof. Let D be a trapped region and assume that $D \cap U \neq \varnothing$. As $\overline{D}$ is compact there is a point $p \in \overline{D}$ where $f\big|_{\overline{D}}$ has its maximum. As f has no critical points the point p is on the boundary ∂D. Setting $r = f(p) = \max_{x \in \overline{D}} f(x)$ we then have $D \subseteq f^{-1}\big[(-\infty, r)\big]$ and ∂D is tangent to

$\partial f^{-1}\big[(-\infty, r)\big]$ at p. But this contradicts the strong maximum principle as $\theta > 0$ on ∂D (so that it is curved outward) and $\theta \le 0$ on $\partial f^{-1}\big[(-\infty, r)\big]$ (so that it is curved inward). This completes the proof. $\square$

As an example on $M = \mathbf{R}^n$ note that the unit sphere bundle $\mathbb{U}(\mathbf{R}^n) = \mathbf{R}^n \times S^{n-1}$ in a natural way. Let $k\colon \mathbf{R}^n \times S^{n-1} \to \mathbf{R}$ be a C^1 function so that for some r_0 we have $k(p,u) \le 1/\|p\|$ for all u and all p with $\|p\| \ge r_0$. Then set $U = \mathbf{R}^n \setminus \overline{B}^n(r_0)$ and let $f(x) = \|x\|$. Then the sets $f^{-1}\big[(-\infty, r)\big]$ are balls of radius r and so the mean curvature of boundaries these is $H = -1/r$. Thus the k-mean curvature satisfies $\theta = H + k \le -1/r + k \le 0$. So f is a barrier for θ. Thus the extremal trapped region, $\mathcal{D}$, is a subset of B^n.

4. Basics about Regularity

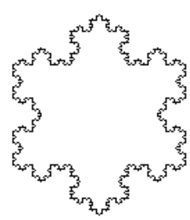
The following summarizes the regularity results that I hope can be gotten cheaply from known regularity results for viscosity solutions [1]. We first isolate the property of boundary point that lets use use this this theory.

4.1**. Definition.** Let M be a smooth manifold and U and open set. Then ∂U is a ***local graph*** (respectively ***local Lipschitz graph*** near p iff there is an open set N containing p which are the domain of smooth coordinates $x^1, \dots, x^n$ so that in these coordinates

$$\partial U \cap N = \{x^n = f(x^1, \dots, x^{n-1})\}$$

for some continuous (respectively Lipschitz) function f. $\square$

4.2. *Remark.* There are domains, D, whose boundaries, ∂D, are imbedded topological hypersurfaces but are not local graphs with respect to any smooth coordinates. The basic example is the domain D bounded by the



Koch snowflake curve. Note however in this example that ∂D is an embedded topological circle and so by the Schönflies Theorem there is a homeomorphism h of $\mathbf{R}^2$ so that $h[D] = B^2 = \{(x,y) : x^2 + y^2 < 1\}$. Therefore ∂D is a local graph with respect to some continuous coordinates. $\square$

Here are two tries at a regularity theorem.

4.3**. Theorem.** *Let D be an open set in the manifold M so that $\theta^{\partial D} = 0$ in the viscosity sense. Assume that ∂D is a local Lipschitz graph near $p \in \partial D$. Then near p the boundary, ∂D, is a smooth hypersurface satisfying $\theta^{\partial D} = 0$ in the classical sense.*

Proof. From the definition of local Lipschitz graph we can assume that we have coordinates $x^1, \dots, x^n$ so that ∂D is given by $x^n = f(x^1, \dots, x^{n-1})$. If we write the mean k-mean curvature on functions $x^n = \varphi(x^1, \dots, x^{n-1})$ as an operator, $\mathcal{M}$, then it will be a quasi-linear elliptic operator and when restricted to functions that satisfy an common Lipschitz condition it will be uniformly elliptic. It is not hard to check that $\theta^{\partial D} = 0$ in the geometric viscosity sense used here implies that $\mathcal{M}(f) = 0$ in the analytic viscosity sense used in [1]. Therefore the regularity theory of [1] implies that f is smooth and then $\mathcal{M}(f) = 0$ in the classical sense. Therefore near p we see that ∂D is a smooth hypersurface satisfying $\theta^{\partial D} = 0$ as required. (This is an oversimplification, as the operator $\mathcal{M}$ may not satisfy all the monotonicity requirements used in [1], but I still think the basic method may work.) $\square$

Here is an idea based on known existence theory and some form of the maximum principle and which does away with the Lipschitz requirement.

4.4. **Theorem.** *Let D be an open set in the manifold M so that $\theta^{\partial D} = 0$ in the viscosity sense. Assume that ∂D is a local graph near $p \in \partial D$. Then near p the boundary, ∂D, is a smooth hypersurface satisfying $\theta^{\partial D} = 0$ in the classical sense.*

Proof. Again near p choose coordinates $x^1, \dots, x^n$ so that ∂D is given by $x^n = f(x^1, \dots, x^{n-1})$ near p. We assume that the domain of these coordinates is $B^{n-1} \times (-1, 1)$, so that $f\colon B^{n-1} \to (-1, 1)$. And we let $\mathcal{M}$ be as in the last proof. For any $x_0 \in B^{n-1}$ let $U \subset B^{n-1}$ containing x_0 and so that the boundary value problem $\mathcal{M}[u] = 0$, $u\big|_{\partial U} = \varphi$ has a unique solution for any continuous $\varphi\colon \partial D \to \mathbf{R}$. Such domains should exist from the theory of quasi-linear equations of mean curvature type [3]. Therefore we can find a solution u so the problem $\mathcal{M}[u] = 0$ and $u\big|_{\partial U} = f\big|_{\partial U}$. Then by uniqueness we should have $u = f$ in U and as u is smooth this implies that f is smooth near x_0. Of course the gap here is that we really only know uniqueness in the class of classical solutions. So we need to generalize uniqueness to the viscosity solutions to quasi-linear operators. What may make this hard is that without a Lipschitz bound, the operator $\mathcal{M}$ is not uniformly elliptic. So this may be harder than I am making it sound.

However in the case that $n = 2$, so that the operator $\mathcal{M}$ is an ordinary differential operator, this should not be hard to push through. $\square$

Based on this and the few example we have so far I will go out on a limb and make:

4.5. **Conjecture.** *Let $D \subset M$ be a domain with $\theta^{\partial D} = 0$ in the viscosity sense. Then there is a closed set $\partial D_{\mathrm{sing}}$ of Hausdorff codimension two in M so that $\partial D \setminus \partial D_{\mathrm{sing}}$ is a smooth hypersurface with $\theta^{\partial D} = 0$ in the classical sense.* $\square$

In this it might be necessary (and natural) to assume that D is regular in the sense of point set topology. That is $\overline{D}^{\circ} = D$. (That is the interior of the closure of D is D.)

5. Things To Think About

Joe,

Here are some directions we might go with this. While I am sure that some of this will look silly latter, I wanted to make a list of problems during my initial enthusiasm as it is going to be moved to a back burner for a while.

Best regards,
Ralph

5.1. **Properties of domains with $\theta > 0$ in the viscosity sense.** Proposition 2.6 shows that an open set with $\theta > 0$ (or $\theta \geq 0$ or $\theta = 0 \dots$) in the viscosity sense has some of the same geometric properties as a domain with smooth boundary and $\theta > 0$ (or $\theta \geq 0$ or $\theta = 0 \dots$) in the classical sense. Work out what what other properties of $\theta > 0$ (or $\theta \geq 0$ or $\theta = 0 \dots$) in the viscosity sense domains. This is probably not of much interest to Riemannian geometers, but might be well received by Lorentzian geometers and general relativists as they are already working with extremal trapped domains and we have a rigorous existence that such domains have $\theta = 0$ in the viscosity sense.

Upside: Should be moderately straightforward and should have an audience. Good way to start to get a feel for viscosity sub and super solutions. Also I have yet to really go through the proofs of the various versions of the Penrose singularity theorem and having to do so would be fun.

Downside: Maybe too easy to be taken seriously by the hard analysis crowd. Also in our hearts we would know this is avoiding the real questions like regularity.

5.2. **Regularity in two dimensions.** I am just about sure that I have a proof that if $D \subset \mathbf{R}^2$ is connected and has $-1 \leq \kappa \leq 0$ on ∂D in the viscosity sense, then D is convex with $C^{1,1}$ boundary. (If D is simply connected then I am about positive this is true.) So I now believe:

5.1. **Conjecture.** *Let $D \subset \mathbf{R}^2$ be a connected open set with ∂D satisfying $-1 \leq \kappa \leq 1$ in the viscosity sense. Then ∂D is a $C^{1,1}$ curve.*

This should be accessible and very likely even fun to work out. It is more general than it looks for if g is any other Riemannian metric on $\mathbf{R}^2$ then inside any set $U \subset \mathbf{R}^2$ with $\overline{U}$ compact there are constants $A, B > 0$, only depending on U and g, so that for any C^2 curve γ in U we have that the curvature κ of γ with respect to the flat metric and the curvature κ^g with respect to g are related by

$$|\kappa| \leq A + B|\kappa^g|.$$

(This calculation should be checked.) So by looking at what happens in the Euclidean metric determined by local coordinates the conjecture would pretty much settle the general regularity question.

Upside: Fun, acessible, has obvious interest.
Downside: Not much that I can see.

5.3. **Regularity in dimensions ≥ 3.** While this is the big question I don't have any real ideas about where to start. I have been meaning to run a seminar out of Caffarelli and Cabrè's book [1] for some time. This may motivate me to get started. We should also look at your idea of showing that for the extremal trapped region $\partial\mathcal{D}$ is rectifiable by getting a bound on the areas ∂D of the trapped surfaces approximating $\partial\mathcal{D}$.

5.4. **Maximum Principle.** Here is what I would like:

5.2**. Conjecture.** *Let M be a complete Riemannian manifold and $k\colon \mathbb{U}(M) \to \mathbf{R}$ a C^1 function so that $k(-u) = -k(u)$ for all $u \in \mathbb{U}(M)$. Let $D_1, D_2 \subset M$ be domains so that the k-mean curvatures satisfy $\theta^{\partial D_i} \geq 0$ in the viscosity sense for $i = 1, 2$. Assume that $D_1 \cap D_2 = \varnothing$, but that there is a point $p \in \partial D_1 \cap \partial D_2$. Then the connected component C of p in ∂D_1 is a smooth embedded hypersurface in M which is also a connected component of ∂D_1 and $\theta = 0$ on C.*

This is going to be hard (and maybe false). But no maximum principle has ever gone unused and this one would be very easy to apply in geometric problems. However getting at least a little bit of a handle on the regularity theory is probably a prerequisite for making any progress here.

References

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