Mathematics 700 Test #2 Name:______________________________

Show your work to get credit. An answer with no work will not get credit.

(1) (15 points) Define or state the following:

(a) A linear map $T\colon V \to V$ is ***diagonalizable*** (where V is a finite dimensional vector space)

(b) The ***adjoint*** of a linear map $S\colon V \to W$ between finite dimensional vector spaces V and W.

(c) ***eigenvalues*** and ***eigenvectors*** of a linear map. (Be sure to be precise about the range and domain).

(d) The ***determinant*** of a linear operator $T\colon V \to V$ on a vector space.

(e) $S^{\perp}$ where S is a non-empty subset of a finite dimensional vector space V.

(2) (10 points) Find the basis of $\mathbf{R}^{2*}$ dual to the basis

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(3) (15 points) Let $\mathcal{P}_2$ be the polynomials of degree ≤ 2 over the real numbers and define a linear map $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ by

$$Tp(x) = p(3x + 2).$$

Find the eigenvectors and values of T.

(4) (10 points) Show directly form the definitions that a linear map $T\colon V \to W$ between finite dimensional vector spaces is injective if and only it its adjoint $T^*\colon W^* \to V^*$ is surjective.

(5) (10 points) Show that if a linear operator $T \colon V \to V$ has eigenvectors v_1, v_2, v_3 with distinct eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, then v_1, v_2, v_3 are linearly independent.

(6) (10 points) Let V be a finite dimensional vector space and W a subspace of V and let $v \in V$ with $v \notin W$. Let $S\colon W \to U$ be a linear map and $u \in U$. Show that there is a linear map $T\colon V \to U$ that extends S and with $Tv = u$.

(7) (10 points) Let V be a vector space and $P\colon V \to V$ a linear map with $P^2 = P$. Show that

$$V = \ker(P) \oplus \mathrm{Image}(P).$$

(8) (10 points) Let V be a finite dimensional vector space and $v_1, v_2, v \in V$ such that for all $f \in V^*$

$$f(v_1) = f(v_2) = 0 \quad \text{implies} \quad f(v) = 0.$$

Show that v is a linear combination of v_1 and v_2.

(9) (10 points) Let $A \in M_{3\times 3}(\mathbf{R})$ be a matrix with characteristic polynomial $x^3 - x$. Then find a diagonal matrix similar to A.