Mathematics 700 Test #2 Name: Solution Key

Show your work to get credit. An answer with no work will not get credit.

(1) (15 points) Define the following:

(a) A linear map $T\colon V \to V$ is ***diagonalizable*** (where V is a finite dimensional vector space).

There is a basis $v_1, \dots, v_n$ of V consisting of eigenvectors of T. **Or:** There is a basis $\mathcal{V} = \{v_1, \dots, v_n\}$ of V such that the matrix $[T]_{\mathcal{V}}$ is a diagonal matrix.

(b) The ***adjoint*** of a linear map $S\colon V \to W$ between finite dimensional vector spaces V and W.

The adjoint is the linear map $S^*\colon W^* \to V^*$ given by

$$S^* g = g \circ S.$$

(Here $\circ$ is function composition so that if $v \in V$ and $g \in W^*$ then $S^*g = g \circ S$ is given by $S^*g(v) = g(Sv)$.

(c) ***eigenvalues*** and ***eigenvectors*** of a linear map. (Be sure to be precise about the range and domain).

Eigenvalues and vectors are only defined for linear maps $T\colon V \to V$ (that is the range and domain are the same). The scalar $\lambda \in \mathbf{F}$ is an eigenvalue iff there is a non-zero vector $v \in V$ such that $Tv = \lambda v$. The vector $v \in V$ is an eigenvector iff it is not the zero vector and $Tv = \lambda v$ for some scalar $\lambda \in \mathbf{F}$.

(d) The ***determinant*** of a linear operator $T\colon V \to V$ on a vector space.

Let $\mathcal{V} = \{v_1, \dots, v_n\}$ be any basis of V and $[T]_{\mathcal{V}}$ the matrix of T with respect to this basis. Then

$$\det T = \det [T]_{\mathcal{V}}.$$

(This is independent of the choice of the basis $\mathcal{V}$ if V.)

(e) $S^\perp$ where S is a non-empty subset of a finite dimensional vector space V.

$$S^\perp = \{f \in V^* : f(x) = 0 \text{ for all } f \in V^*\}.$$

That is $W^\perp$ is the set of linear functional that vanish on all elements of S.

(2) (10 points) Find the basis of $\mathbf{R}^{2*}$ dual to the basis

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Solution: We know that the basis of $\mathbf{F}^{n*}$ dual to a basis $v_1, \dots, v_n$ of $\mathbf{F}^n$ is made up of the rows of the inverse of the matrix $\begin{bmatrix} v_1, v_2, \dots, v_n \end{bmatrix}$ with columns $v_1, \dots, v_n$. In the present case

$$\begin{bmatrix} v_1, v_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}.$$

The inverse is

$$\begin{bmatrix} v_1, v_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 5 - 3 \cdot 2} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

Therefore the dual basis to v_1 and v_2 is

$$f_1 = \begin{bmatrix} -5, 3 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 2, -1 \end{bmatrix}.$$

Or in functional notation

$$f_1 \begin{bmatrix} x \\ y \end{bmatrix} = -5x + 3y, \quad f_2 \begin{bmatrix} x \\ y \end{bmatrix} = 2x - y.$$

(3) (15 points) Let $\mathcal{P}_2$ be the polynomials of degree ≤ 2 over the real numbers and define a linear map $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ by

$$Tp(x) = p(3x+2).$$

Find the eigenvectors and values of T.

Solution: First we find the matrix of T in some basis of $\mathcal{P}_2$. The natural choice is the basis $\mathcal{B} := \{1, x, x^2\}$. In this basis

$$T1 = 1 \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad Tx = 3x + 2 \sim \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad Tx^2 = (3x+2)^2 = 9x^2 + 12x + 4 \sim \begin{bmatrix} 4 \\ 12 \\ 9 \end{bmatrix}.$$

The matrix A of T in this basis has these vectors as columns. That is

$$A := \big[T\big]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix}.$$

Using that the determinant of an upper triangular matrix is the product of the diagonal elemnts we see that the characteristic polynomial of T is

$$\operatorname{char}_T(x) = \det(xI - A) = (x-1)(x-3)(x-9).$$

and that its roots are 1, 3 and 9. Thus the eigenvalues of T (which are the same as the eigenvaluse of A) are $\lambda = 1, 3, 9$. We now find the eigenvectors of A corresponding to these eigenvalues.

For $\lambda = 1$ we want a non-zero vector in the kernel of $I - A = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 12 \\ 0 & 0 & 8 \end{bmatrix}$. This leads to the system for x, y, z

$$2y + 4z = 0, \quad 2y + 12z = 0, \quad 8z = 0.$$

We want any non-zero solution and $x = 1$, $y = z = 0$ works. Thus we get the eigenvector

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 1$.

For $\lambda = 3$ we want a non-zero vector in the kernel of $I - 3A = \begin{bmatrix} -2 & 2 & 4 \\ 0 & 0 & 12 \\ 0 & 0 & 6 \end{bmatrix}$. This leads to the system

$$-2x + 2y + 4z = 0, \quad 12z = 0, \quad 4x = 0.$$

A non-zero solution to this is $x = y = 1$, $z = 0$, which gives the eigenvector

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 3$.

For For $\lambda = 9$ we want a non-zero vector in the kernel of $I - 9A = \begin{bmatrix} -8 & 2 & 4 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \end{bmatrix}$. This leads to the system

$$-8x + 2y + 4z = 0, \quad -6x + 12y = 0, \quad 0 = 0.$$

A non-zero solution is $x = 1$, $y = 2$, $z = 1$, which gives the eigenvector

$$v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

However we are not done. This gives the eigenvectors of the matrix A, but we are looking for the eigenvectors of the linear operator T.

For $\lambda = 1$ the element for $\mathcal{P}_3$ corresponding to v_1 is $p_1 = 1$, for $\lambda = 3$ the element of $\mathcal{P}_3$ corresponding to v_2 is $1+x$ and for $\lambda = 9$ the element corresponding to v_3 is $1+2x+x^2 = (1+x^2)$. In summary the eigenvalues of T are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 9$ with cooresponding eigenvectors

$$p_1(x) = 1, \quad p_2(x) = 1 + x, \quad p_3(x) = 1 + 2x + x^2 = (1 + x)^2.$$

CHECK: We just compute

$$\begin{aligned} Tp_1(x) &= p_1(3x + 2) = 1 = 1p_1(x), \\ Tp_2(x) &= p_2(3x + 2) = 3x + 2 + 1 = 3(x + 1) = 3p_2(x), \\ Tp_3(x) &= p_3(3x + 2) = (3x + 2 + 1)^2 = (3(x + 1))^2 = 9(x + 1)^2 = 9p_2(x). \end{aligned}$$

(4) (10 points) Show directly from the definitions that a linear map $T\colon V \to W$ between finite dimensional vector spaces is injective if and only it its adjoint $T^*\colon W^* \to V^*$ is surjective.

Solution 1: First assume that T^* is surjective. Then to show T is injective it is enough to show that $\ker T = \{0\}$. That is it is enough to show that $Tv = 0$ implies $v = 0$. Let $f \in V^*$. As T^* is surjective there is a $g \in W^*$ with $f = T^*g$. Therefore

$$\langle v, f\rangle = \langle v, T^*g\rangle = \langle Tv, g\rangle = \langle 0, g\rangle = 0.$$

This holds for all $f \in V^*$ so $v = 0$. This show $\ker T = \{0\}$ and thus that T is injective.

Conversely assume that T is injective. This implies that $\ker T = \{0\}$. If $\operatorname{Image} T^*$ is not all of V^* then, $\operatorname{Image} T^*$ is a proper subspace of V^* and so there is some $f_0 \notin \operatorname{Image} T^*$. But there we can separate f_0 from the subspace $\operatorname{Image} T^*$ by an evaluation. That is there is a $v_0 \in V$ with $\langle v_0, f_0\rangle = f_0(v_0) = 1$ and $\langle v_0, f_0\rangle = f_0(v_0) = 0$ for all $f \in \operatorname{Image} T^*$. If $g \in W^*$ then $T^*g \in \operatorname{Image} T^*$ and thus

$$0 = \langle v_0, T^*g\rangle = \langle Tv_0, g\rangle.$$

Thus $\langle Tv_0, g\rangle = 0$ For all $g \in W^*$. This implies $Tv_0 = 0$. But $\langle v_0, f_0\rangle = 1$ implies that $v_0 \neq 0$. This contradicts that $\ker T = \{0\}$. Thus T^* is surjective. *done.*

Solution 2: (This is due to Wally, though others had an equivalent but slightly less elegant version of the some proof.)

Let $\dim V = n$. Then

$$\begin{aligned} T \text{ is injective} &\iff \operatorname{nullity}(T) = 0 \\ &\iff \operatorname{rank}(T) = n && \text{(by rank plus nullity theorem)} \\ &\iff \operatorname{rank}(T^*) = n && \text{(as } \operatorname{rank}(T) = \operatorname{rank}(T^*)) \\ &\iff \dim \operatorname{Image}(T^*) = n \\ &\iff \operatorname{Image}(T^*) = V^* && \text{(as } \dim V^* = \dim V = n) \\ &\iff T^* \text{ in surjective.} \end{aligned}$$

Remark: What I had in mind with the phrase "directly from the definitions" was that I did not want you to use that $\ker(T)^\perp = \operatorname{Image}(T^*)$ which makes the proof a one liner (as $\ker(T)^\perp = \operatorname{Image}(T^*) = V^*$ if and only if $\ker(T) = \{0\}$). I Should have been more explicit about what could and could not be used.

(5) (10 points) Show that if a linear operator $T\colon V \to V$ has eigenvectors v_1, v_2, v_3 with distinct eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, then v_1, v_2, v_3 are linearly independent.

Solution: Let $c_1, c_2, c_3 \in \mathbf{F}$ be scalars such that

$$c_1v_1 + c_2v_2 + c_3v_3 = 0. \tag{1}$$

As v_1, v_2, v_3 are eigenvectors we have

$$Tv_1 = \lambda_1v_1, \quad Tv_2 = \lambda_2v_2, \quad Tv_3 = \lambda_3v_3. \tag{2}$$

Now apply T to (1) and use (2)

$$0 = T0 = c_1Tv_1 + c_2Tv_2 + c_3Tv_3 = c_1\lambda_1v_1 + c_2\lambda_2v_2 + c_3\lambda_3v_3. \tag{3}$$

Multiply (1) by λ_2.

(4) $$c_1\lambda_3 v_1 + c_2\lambda_3 v_2 + c_3\lambda_3 v_3 = 0$$

If (4) is subtracted from (3) the v_3 term cancels out and we are left with

(5) $$c_1(\lambda_3 - \lambda_1)v_1 + c_2(\lambda_3 - \lambda_2)v_2 = 0.$$

Now do the same trick again. Applying T to both sides of (5) and using (2) gives

$$c_1(\lambda_3 - \lambda_1)\lambda_1 v_1 + c_2(\lambda_3 - \lambda_2)\lambda_2 v_2 = 0.$$

Multiplying (5) by λ_2 gives

$$c_1(\lambda_3 - \lambda_1)\lambda_2 v_1 + c_2(\lambda_3 - \lambda_2)\lambda_2 v_2 = 0.$$

Subtracting these gives

$$c_1(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)v_1 = 0.$$

As $v_1 \neq 0$ (as it is an eigenvector) and $\lambda_1, \lambda_2, \lambda_3$ are distinct this implies $c_1 = 0$. Using $c_1 = 0$ in (5) gives

$$c_2(\lambda_3 - \lambda_2)v_2 = 0.$$

This implies $c_2 = 0$. Now using $c_1 = c_2 = 0$ in (1) implies $c_3 = 0$. Thus we have shown that whenever (1) holds that $c_1 = c_2 = c_3 = 0$. Therefore v_1, v_2, v_3 are linearly independent. *done.*

(6) (10 points) Let V be a finite dimensional vector space and W a subspace of V and let $v \in V$ with $v \notin W$. Let $S\colon W \to U$ be a linear map and $u \in U$. Show that there is a linear map $T\colon V \to U$ that extends S and with $Tv = u$.

Solution: Let $k = \dim W$ and $n = \dim V$. Choose a basis $v_1, v_2, \ldots, v_k$ of W. Then $v \notin W = \operatorname{Span}\{v_1, \ldots, v_k\}$ implies that $\{v_1, , \ldots, v_k, v\}$ is linearly independent. So let $v_{k+1} = v$ and extend the linearly independent set $v_1, \ldots, v_k, v_{k+1}$ to a basis $v_1, , \ldots, v_n$ of V. By the basic existence theorem for linear maps there is a linear map $T\colon V \to U$ such that

$$Tv_i = \begin{cases} Sv_i, & 1 \leq i \leq k; \\ u, & i = k+1; \\ 0, & k+2 \leq i \leq n. \end{cases}$$

Then $Tv_i = Sv_i$ for $1 \leq i \leq k$ and thus $T\big|_W$ and S agree on a basis of W. Therefore $T\big|_W = S$ and thus T extends S. Also $Tv = Tv_{k+1} = u$. *done.*

(7) (10 points) Let V be a vector space and $P\colon V \to V$ a linear map with $P^2 = P$. Show that

$$V = \ker(P) \oplus \operatorname{Image}(P).$$

Solution: We need to show that $\ker(P) + \operatorname{Image}(P) = V$ and $\ker(P) \cap \operatorname{Image}(P) = \{0\}$. Note for any $v \in V$ that

$$P(v - Pv) = Pv - P^2 v = Pv - Pv = 0$$

as $P^2 = P$. Thus

$$(v - Pv) \in \ker(P) \quad \text{for all} \quad v \in V.$$

Now for any $v \in V$

$$v = (v - Pv) + Pv.$$

Clearly $Pv \in \operatorname{Image}(P)$ and we have just seen $(v - Pv) \in \ker(V)$. Thus every element of V is a sum of an element of $\ker(P)$ and an element of $\operatorname{Image}(P)$, whence $V = \ker(P) + \operatorname{Image}(P)$.

It remains to show that $\ker(P) \cap \operatorname{Image}(P) = \{0\}$. Let $v \in \ker(P) \cap \operatorname{Image}(P)$. Then $Pv = 0$ as $v \in \ker(P)$. As $v \in \operatorname{Image}(P)$ there a $v' \in V$ with $v = Pv'$. But then, using $P^2 = P$ and $Pv = 0$,

$$v = Pv' = P^2 v' = PPv' = Pv = 0.$$

Thus if $v \in \ker(P) \cap \operatorname{Image}(P)$, then $v = 0$. Therefore $\ker(P) \cap \operatorname{Image}(P) = \{0\}$. *done.*

(8) (10 points) Let V be a finite dimensional vector space and $v_1, v_2, v \in V$ such that for all $f \in V^*$

$$f(v_1) = f(v_2) = 0 \quad \text{implies} \quad f(v) = 0.$$

Show that v is a linear combination of v_1 and v_2.

Solution 1: We wish to show that $v \in \operatorname{Span}\{v_1, v_2\}$. Assume, toward a contradiction, that $v \notin \operatorname{Span}\{v_1, v_2\}$. Then as $\operatorname{Span}\{v_1, v_2\}$ is a subspace of V there is a linear functional $f \in V^*$

that separates v from $\text{Span}\{v_1, v_2\}$. That is $f(v) = 1$, but $f(w) = 0$ for all $w \in \text{Span}\{v_1, v_2\}$. As $v_1, v_2 \in \text{Span}\{v_1, v_2\}$ we have

$$f(v_1) = f(v_2) = 0, \quad \text{but} \quad f(v) = 1.$$

This clearly contradicts our assumption that $f(v_1) = f(v_2) = 0$ implies $f(v) = 0$ and we are *done.*

Solution 2: The hypothesis is that $f(v) = 0$ for any $f \in V^*$ with $f(v_1) = f(v_2) = 0$. But the set of $f \in V^*$ with $f(v_1) = f(v_2) = 0$ is $\{v_1, v_2\}^\perp$. Therefore the hypothesis can be restated as $f(v) = 0$ for all $f \in \{v_1, v_2\}^\perp$. But this is just the definition of $v \in \left(\{v_1, v_2\}^\perp\right)^\circ$. So we have

$$v \in \left(\{v_1, v_2\}^\perp\right)^\circ = \text{Span}\{v_1, v_2\}$$

as $\left(S^\perp\right)^\circ = \text{Span}(S)$ for any non-empety subset S of V. *done.*

(9) (10 points) Let $A \in M_{3\times 3}(\mathbf{R})$ be a matrix with characteristic polynomial $x^3 - x$. Then find a diagonal matrix similar to A.

Solution: (This is basically due to Mindy, and is a bit more informative than what I did.) Note that $\text{char}_A(x) = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$. Thus the eigenvalues of A are $-1, 0, 1$. Let v_1 be an eigenvector for $\lambda = -1$, v_2 an eigenvector for $\lambda = 0$ and v_3 an eigenvector for $\lambda = 3$. As $-1, 0, 1$ are distinct, the eigenvectors v_1, v_2, v_3 are linearly indepent. Let $P = \left[v_1, v_2, v_3\right]$ be the matrix with v_1, v_2, v_3 as columns. Then we have a theorem that says that

$$P^{-1}AP = \text{diag}(-1, 0, 1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This shows that A is similar to $\text{diag}(-1, 0, 1)$ *done.*