

**LINEAR ALGEBRA QUESTIONS FROM THE ADMISSION TO CANDIDACY
EXAM**

The following is a more or less complete list of the linear algebra questions that have appeared on the admission to candidacy exam for the last fifteen years. Some questions have been reworded a little.

JANUARY 1984

1. Let V be a finite-dimensional vector space and let T be a linear operator on V . Suppose that T commutes with every diagonalizable linear operator on V . Prove that T is a scalar multiple of the identity operator.
2. Let V and W be vector spaces and let T be a linear operator from V into W . Suppose that V is finite-dimensional. Prove $\text{rank}(T) + \text{nullity}(T) = \dim V$.
3. Let A and B be $n \times n$ matrices over a field \mathbf{F} .
 - (a) Prove that if A or B is nonsingular, then AB is similar to BA .
 - (b) Show that there exist matrices A and B so that AB is *not* similar to BA .
 - (c) What can you deduce about the eigenvalues of AB and BA ? Prove your answer.
4. Let $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$, where D and G are $n \times n$ matrices. If $DF = FD$ prove that $\det A = \det(DG - FE)$.
5. If \mathbf{F} is a field, prove that every ideal in $\mathbf{F}[x]$ is principal.

AUGUST 1984

1. Let V be a finite dimensional vector space. Can V have three distinct proper subspaces W_0 , W_1 and W_2 such that $W_0 \subseteq W_1$, $W_0 + W_2 = V$, and $W_1 \cap W_2 = \{0\}$?
2. Let n be a positive integer. Define

$$G = \{A : A \text{ is an } n \times n \text{ matrix with only integer entries and } \det A \in \{-1, +1\}\},$$

$$H = \{A : A \text{ is an invertible } n \times n \text{ matrix and both } A \text{ and } A^{-1} \text{ have only integer entries}\}.$$
 Prove $G = H$.
3. Let V be the vector space over \mathbf{R} of all $n \times n$ matrices with entries from \mathbf{R} .
 - (a) Prove that $\{I, A, A^2, \dots, A^n\}$ is linearly dependent for all $A \in V$.
 - (b) Let $A \in V$. Prove that A is invertible if and only if I belongs to the span of $\{A, A^2, \dots, A^n\}$.
4. Is every $n \times n$ matrix over the field of complex numbers similar to a matrix of the form $D + N$, where D is a diagonal matrix, $N^{n-1} = 0$, and $DN = ND$? Why?

JANUARY 1985

1. (a) Let V and W be vector spaces and let T be a linear operator from V into W . Suppose that V is finite-dimensional. Prove $\text{rank}(T) + \text{nullity}(T) = \dim V$.
 (b) Let $T \in L(V, V)$, where V is a finite dimensional vector space. (For a linear operator S denote by $\mathcal{N}(S)$ the null space and by $\mathcal{R}(S)$ the range of S .)
 - (i) Prove there is a least natural number k such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1}) = \mathcal{N}(T^{k+2}) \dots$. Use this k in the rest of this problem.
 - (ii) Prove that $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1}) = \mathcal{R}(T^{k+2}) \dots$.
 - (iii) Prove that $\mathcal{N}(T^k) \cap \mathcal{R}(T^k) = \{0\}$.
 - (iv) Prove that for each $\alpha \in V$ there is exactly one vector in $\alpha_1 \in \mathcal{N}(T^k)$ and exactly one vector $\alpha_2 \in \mathcal{R}(T^k)$ such that $\alpha = \alpha_1 + \alpha_2$.
2. Let \mathbf{F} be a field of characteristic 0 and let

$$W = \left\{ A = [a_{ij}] \in \mathbf{F}^{n \times n} : \text{tr}(A) = \sum_{i=1}^n a_{ii} = 0 \right\}.$$

For $i, j = 1, \dots, n$ with $i \neq j$, let E_{ij} be the $n \times n$ matrix with (i, j) -th entry 1 and all the remaining entries 0. For $i = 2, \dots, n$ let E_i be the $n \times n$ matrix with $(1, 1)$ entry -1 , (i, i) -th entry $+1$, and all remaining entries 0. Let

$$S = \{E_{ij} : i, j = 1, \dots, n \text{ and } i \neq j\} \cup \{E_i : i = 2, \dots, n\}.$$

[NOTE: You can assume, without proof, that S is a linearly independent subset of $\mathbf{F}^{n \times n}$.]

- (a) Prove that W is a subspace of $\mathbf{F}^{n \times n}$ and that $W = \text{span}(S)$. What is the dimension of W ?
- (b) Suppose that f is a linear functional on $\mathbf{F}^{n \times n}$ such that
 - (i) $f(AB) = f(BA)$, for all $A, B \in \mathbf{F}^{n \times n}$.
 - (ii) $f(I) = n$, where I is the identity matrix in $\mathbf{F}^{n \times n}$.

Prove that $f(A) = \text{tr}(A)$ for all $A \in \mathbf{F}^{n \times n}$.

AUGUST 1985

1. Let V be a vector space over \mathbf{C} . Suppose that f and g are linear functionals on V such that the functional

$$h(\alpha) = f(\alpha)g(\alpha) \quad \text{for all } \alpha \in V$$

is linear. Show that either $f = 0$ or $g = 0$.

2. Let C be a 2×2 matrix over a field \mathbf{F} . Prove: There exists matrices $C = AB - BA$ if and only if $\text{tr}(C) = 0$.
3. Prove that if A and B are $n \times n$ matrices from \mathbf{C} and $AB = BA$, then A and B have a common eigenvector.

JANUARY 1986

1. Let \mathbf{F} be a field and let V be a finite dimensional vector space over \mathbf{F} . Let $T \in L(V, V)$. If c is an eigenvalue of T , then prove there is a nonzero linear functional f in $L(V, \mathbf{F})$ such that $T^*f = cf$. (Recall that $T^*f = fT$ by definition.)
2. Let \mathbf{F} be a field, $n \geq 2$ be an integer, and let V be the vector space of $n \times n$ matrices over \mathbf{F} . Let A be a fixed element of V and let $T \in L(V, V)$ be defined by $T(B) = AB$.
 - (a) Prove that T and A have the same minimal polynomial.
 - (b) If A is diagonalizable, prove, or disprove by counterexample, that T is diagonalizable.
 - (c) Do A and T have the same characteristic polynomial? Why or why not?
3. Let M and N be 6×6 matrices over \mathbf{C} , both having minimal polynomial x^3 .
 - (a) Prove that M and N are similar if and only if they have the same rank.
 - (b) Give a counterexample to show that the statement is false if 6 is replaced by 7.

AUGUST 1986

1. Give an example of two 4×4 matrices that are not similar but that have the same minimal polynomial.
2. Let (a_1, a_2, \dots, a_n) be a nonzero vector in the real n -dimensional space \mathbf{R}^n and let P be the hyperplane

$$\left\{ (x_1, x_2, \dots, x_n) \in \mathbf{R}^n : \sum_{i=1}^n a_i x_i = 0 \right\}.$$

Find the matrix that gives the reflection across P .

JANUARY 1987

1. Let V and W be finite-dimensional vector spaces and let $T : V \rightarrow W$ be a linear transformation. Prove that there exists a basis \mathcal{A} of V and a basis \mathcal{B} of W so that the matrix $[T]_{\mathcal{A}, \mathcal{B}}$ has the block form $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.
2. Let V be a finite-dimensional vector space and let T be a diagonalizable linear operator on V . Prove that if W is a T -invariant subspace then the restriction of T to W is also diagonalizable.
3. Let T be a linear operator on a finite-dimensional vector. Show that if T has no cyclic vector then, then there exists an operator U on V that commutes with T but is *not* a polynomial in T .

AUGUST 1987

1. Exhibit two real matrices with no real eigenvalues which have the same characteristic polynomial and the same minimal polynomial but are not similar.
2. Let V be a vector space, not necessarily finite-dimensional. Can V have three distinct proper subspaces A , B , and C , such that $A \subset B$, $A + C = V$, and $B \cap C = \{0\}$?
3. Compute the minimal and characteristic polynomials of the following matrix. Is it diagonalizable?

$$\begin{bmatrix} 5 & -2 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

AUGUST 1988

1. (a) Prove that if A and B are linear transformations on an n -dimensional vector space with $AB = 0$, then $r(A) + r(B) \leq n$ where $r(\cdot)$ denotes rank.
 (b) For each linear transformation A on an n -dimensional vector space, prove that there exists a linear transformation B such that $AB = 0$ and $r(A) + r(B) = n$.
2. (a) Prove that if A is a linear transformation such that $A^2(I - A) = A(I - A)^2 = 0$, then A is a projection.
 (b) Find a non-zero linear transformation so that $A^2(I - A) = 0$ but A is *not* a projection.
3. If S is an m -dimensional vector space of an n -dimensional vector space V , prove that S° , the annihilator of S , is an $(n - m)$ -dimensional subspace of V^* .
4. Let A be the 4×4 real matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

- (a) Determine the rational canonical form of A .
- (b) Determine the Jordan canonical form of A .

JANUARY 1989

1. Let T be the linear operator on \mathbf{R}^3 which is represented by

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

in the standard basis. Find matrices B and C which represent respectively, in the standard basis, a diagonalizable linear operator D and a nilpotent linear operator N such that $T = D + N$ and $DN = ND$.

2. Suppose T is a linear operator on \mathbf{R}^5 represented in some basis by a diagonal matrix with entries $-1, -1, 5, 5, 5$ on the main diagonal.
 (a) Explain why T can not have a cyclic vector.
 (b) Find k and the invariant factors $p_i = p_{\alpha_i}$ in the cyclic decomposition $\mathbf{R}^5 = \bigoplus_{i=1}^k Z(\alpha_i; T)$.
 (c) Write the rational canonical form for T .
3. Suppose that V is an n -dimensional vector space and T is a linear map on V of rank 1. Prove that T is nilpotent or diagonalizable.

AUGUST 1989

1. Let M denote an $m \times n$ matrix with entries in a field. Prove that

$$\begin{aligned} & \text{the maximum number of linearly independent rows of } M \\ &= \text{the maximum number of linearly independent columns of } M \end{aligned}$$

(Do not assume that $\text{rank } M = \text{rank } M^t$.)

- Prove the Cayley-Hamilton Theorem, using only basic properties of determinants.
- Let V be a finite-dimensional vector space. Prove there a linear operator T on V is invertible if and only if the constant term in the minimal polynomial for T is non-zero.
- (a) Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. Find a matrix T (with entries in \mathbf{C}) such that $T^{-1}MT$ is diagonal, or prove that such a matrix does not exist.
 (b) Find a matrix whose minimal polynomial is $x^2(x-1)^2$, whose characteristic polynomial is $x^4(x-1)^3$ and whose rank is 4.
- Suppose A and B are linear operators on the same finite-dimensional vector space V . Prove that AB and BA have the same characteristic values.
- Let M denote an $n \times n$ matrix with entries in a field \mathbf{F} . Prove that there is an $n \times n$ matrix B with entries in \mathbf{F} so that $\det(M + tB) \neq 0$ for every non-zero $t \in \mathbf{F}$.

JANUARY 1990

- Let W_1 and W_2 be subspaces of the finite dimensional vector space V . Record and prove a formula which relates $\dim W_1$, $\dim W_2$, $\dim(W_1 + W_2)$, $\dim(W_1 \cap W_2)$.
- Let M be a symmetric $n \times n$ matrix with real number entries. Prove that there is an $n \times n$ matrix N with real entries such that $N^3 = M$.
- TRUE OR FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample.) If two nilpotent matrices have the same rank, the same minimal polynomial and the same characteristic polynomial, then they are similar.

AUGUST 1990

- Suppose that $T : V \rightarrow W$ is a injective linear transformation over a field \mathbf{F} . Prove that $T^* : W^* \rightarrow V^*$ is surjective. (Recall that $V^* = L(V, \mathbf{F})$ is the vector space of linear transformations from V to \mathbf{F} .)
- If M is the $n \times n$ matrix

$$M = \begin{bmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x \end{bmatrix}$$

then prove that $\det M = [x + (n-1)a](x-a)^{n-1}$.

- Suppose that T is a linear operator on a finite dimensional vector space V over a field \mathbf{F} . Prove that T has a cyclic vector if and only if

$$\{U \in L(V, V) : TU = UT\} = \{f(T) : f \in \mathbf{F}[x]\}.$$

- Let $T : \mathbf{R}^4 \rightarrow \mathbf{R}^4$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_1, -2x_2 - x_3 - 4x_4, 4x_2 + x_3)$$

- Compute the characteristic polynomial of T .
- Compute the minimal polynomial of T .
- The vector space \mathbf{R}^4 is the direct sum of two proper T -invariant subspaces. Exhibit a basis for one of these subspaces.

JANUARY 1991

- Let V , W , and Z be finite dimensional vector spaces over the field \mathbf{F} and let $f : V \rightarrow W$ and $g : W \rightarrow Z$ be linear transformations. Prove that

$$\text{nullity}(g \circ f) \leq \text{nullity}(f) + \text{nullity}(g)$$

2. Prove that

$$\det \begin{bmatrix} A & 0 & 0 \\ B & C & D \\ 0 & 0 & E \end{bmatrix} = \det A \det C \det E$$

where A, B, C, D and E are all square matrices.

3. Let A and B be $n \times n$ matrices with entries on the field \mathbf{F} such that $A^{n-1} \neq 0$, $B^{n-1} \neq 0$, and $A^n = B^n = 0$. Prove that A and B are similar, or show, with a counterexample, that A and B are not necessarily similar.

AUGUST 1991

- Let A and B be $n \times n$ matrices with entries from \mathbf{R} . Suppose that A and B are similar over \mathbf{C} . Prove that they are similar over \mathbf{R} .
- Let A be an $n \times n$ with entries from the field \mathbf{F} . Suppose that $A^2 = A$. Prove that the rank of A is equal to the trace of A .
- TRUE OR FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample.) Let W be a vector space over a field \mathbf{F} and let $\theta : V \rightarrow V'$ be a fixed surjective transformation. If $g : W \rightarrow V'$ is a linear transformation then there is linear transformation $h : W \rightarrow V$ such that $\theta \circ h = g$.

JANUARY 1992

- Let V be a finite dimensional vector space and $A \in L(V, V)$.
 - Prove that there exists an integer k such that $\ker A^k = \ker A^{k+1} = \dots$
 - Prove that there exists an integer k such that $V = \ker A^k \oplus \text{image } A^k$.
- Let V be the vector space of $n \times n$ matrices over a field \mathbf{F} , and let $T : V \rightarrow V^*$ be defined by $T(A)(B) = \text{tr}(A^t B)$. Prove that T is an isomorphism.
- Let A be an $n \times n$ matrix and $A^k = 0$ for some k . Show that $\det(A + I) = 1$.
- Let V be a finite dimensional vector space over a field \mathbf{F} , and T a linear operator on V . Suppose the minimal and characteristic polynomials of T are the same power of an irreducible polynomial $p(x)$. Show that no non-trivial T -invariant subspace of V has a T -invariant complement.

AUGUST 1992

- Let V be the vector space of all $n \times n$ matrices over a field \mathbf{F} , and let B be a fixed $n \times n$ matrix that is not of the form cI . Define a linear operator T on V by $T(A) = AB - BA$. Exhibit a not-zero element in the kernel of the transpose of T .
- Let V be a finite dimensional vector space over a field \mathbf{F} and suppose that S and T are triangulable operators on V . If $ST = TS$ prove that S and T have an eigenvector in common.
- Let A be an $n \times n$ matrix over \mathbf{C} . If $\text{trace } A^i = 0$ for all $i > 0$, prove that A is nilpotent.

JANUARY 1993

- Let V be a finite dimensional vector space over a field \mathbf{F} , and let T be a linear operator on V so that $\text{rank}(T) = \text{rank}(T^2)$. Prove that V is the direct sum of the range of T and the null space of T .
- Let V be the vector space of all $n \times n$ matrices over a field \mathbf{F} , and suppose that A is in V . Define $T : V \rightarrow V$ by $T(AB) = AB$. Prove that A and T have the same characteristic values.
- Let A and B be $n \times n$ matrices over the complex numbers.
 - Show that AB and BA have the same characteristic values.
 - Are AB and BA similar matrices?
- Let V be a finite dimensional vector space over a field of characteristic 0, and T be a linear operator on V so that $\text{tr}(T^k) = 0$ for all $k \geq 1$, where $\text{tr}(\cdot)$ denotes the trace function. Prove that T is a nilpotent linear map.
- Let $A = [a_{ij}]$ be an $n \times n$ matrix over the field of complex numbers such that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{for } i = 1, \dots, n.$$

Then show that $\det A \neq 0$. (\det denotes the determinant.)

6. Let A be an $n \times n$ matrix, and let $\text{adj}(A)$ denote the adjoint of A . Prove the rank of $\text{adj}(A)$ is either 0, 1, or n .

AUGUST 1993

1. Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}$$

- (a) Determine the rational canonical form of A .
(b) Determine the Jordan canonical form of A .
2. If

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then prove that there does not exist a matrix with $N^2 = A$.

3. Let A be a real $n \times n$ matrix which is symmetric, i.e. $A^t = A$. Prove that A is diagonalizable.
4. Give an example of two nilpotent matrices A and B such that
- (a) A is not similar to B ,
(b) A and B have the same characteristic polynomial,
(c) A and B have the same minimal polynomial, and
(d) A and B have the same rank.

JANUARY 1994

1. Let A be an $n \times n$ matrix over a field \mathbf{F} . Show that \mathbf{F}^n is the direct sum of the null space and the range of A if and only if A and A^2 have the same rank.
2. Let A and B be $n \times n$ matrices over a field \mathbf{F} .
- (a) Show AB and BA have the same eigenvalues.
(b) Is AB similar to BA ? (Justify your answer).
3. Given an exact sequence of finite-dimensional vector spaces

$$0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} 0$$

that is the range of T_i is equal to the null space of T_{i+1} , for all i . What is the value of

$\sum_{i=1}^n (-1)^i \dim(V_i)$? (Justify your answer).

4. Let \mathbf{F} be a field with q elements and V be a n -dimensional vector space over \mathbf{F} .
- (a) Find the number of elements in V .
(b) Find the number of bases of V .
(c) Find the number of invertible linear operators on V .
5. Let A and B be $n \times n$ matrices over a field \mathbf{F} . Suppose that A and B have the same trace and the same minimal polynomial of degree $n - 1$. Is A similar to B ? (Justify your answer.)
6. Let $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{ij} = 1$ for all i and j . Find its characteristic and minimal polynomial.

AUGUST 1994

1. Give an example of a matrix with real entries whose characteristic polynomial is $x^5 - x^4 + x^2 - 3x + 1$.
2. TRUE or FALSE. (If true prove it. If false give a counterexample.) Let A and B be $n \times n$ matrices with minimal polynomial x^4 . If $\text{rank } A = \text{rank } B$, and $\text{rank } A^2 = \text{rank } B^2$, then A and B are similar.

3. Suppose that T is a linear operator on a finite-dimensional vector space V over a field \mathbf{F} . Prove that the characteristic polynomial of T is equal to the minimal polynomial of T if and only if

$$\{U \in L(V, V) : TU = UT\} = \{f(T) : f \in \mathbf{F}[x]\}.$$

JANUARY 1995

- (a) Prove that if A and B are 3×3 matrices over a field \mathbf{F} , a necessary and sufficient condition that A and B be similar over \mathbf{F} is that they have the same characteristic and the same minimal polynomial.
(b) Give an example to show this is not true for 4×4 matrices.
- Let V be the vector space of $n \times n$ matrices over a field. Assume that f is a linear functional on V so that $f(AB) = f(BA)$ for all $A, B \in V$, and $f(I) = n$. Prove that f is the trace functional.
- Suppose that N is a 4×4 nilpotent matrix over \mathbf{F} with minimal polynomial x^2 . What are the possible rational canonical forms for n ?
- Let A and B be $n \times n$ matrices over a field \mathbf{F} . Prove that AB and BA have the same characteristic polynomial.
- Suppose that \mathbf{V} is an n -dimensional vector space over \mathbf{F} , and T is a linear operator on \mathbf{V} which has n distinct characteristic values. Prove that if S is a linear operator on \mathbf{V} that commutes with T , then S is a polynomial in T .

AUGUST 1995

- Let A and B be $n \times n$ matrices over a field \mathbf{F} . Show that AB and BA have the same characteristic values in \mathbf{F} .
- Let P and Q be real $n \times n$ matrices so that $P + Q = I$ and $\text{rank}(P) + \text{rank}(Q) = n$. Prove that P and Q are projections. (HINT: Show that if $Px = Qy$ for some vectors x and y , then $Px = Qy = 0$.)
- Suppose that A is an $n \times n$ real, invertible matrix. Show that A^{-1} can be expressed as a polynomial in A with real coefficients and with degree at most $n - 1$.
- Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Determine the rational canonical form and the Jordan canonical form of A .

- (a) Give an example of two 4×4 nilpotent matrices which have the same minimal polynomial but are not similar.
(b) Explain why 4 is the smallest value that can be chosen for the example in part (a), i.e. if $n \leq 3$, any two nilpotent matrices with the same minimal polynomial are similar.

JANUARY 1996

- Let \mathcal{P}_3 be the vector space of all polynomials with coefficients from \mathbf{R} and of degree at most 3. Define a linear $T : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ by $(Tf)(x) = f(2x - 6)$. Is T diagonalizable? Explain why.
- Let R be the ring of $n \times n$ matrices over the real numbers. Show that R does not have any two sided ideals other than R and $\{0\}$.
- Let V be a finite dimensional vector space and $A : V \rightarrow V$ a linear map. Suppose that $V = U \oplus W$ is a direct sum decomposition of V into subspaces invariant under A . Let V^* be the dual space of V and let $A^t : V^* \rightarrow V^*$ be the transpose of A .
(a) Show that V^* has a direct sum decomposition $V^* = X \oplus Y$ so that $\dim X = \dim U$ and $\dim Y = \dim W$ and both X and Y are invariant under A^t .
(b) Using part (a), or otherwise, prove that A and A^t are similar.

AUGUST 1996

- Consider a linear operator on the space of 3×3 matrices defined by $S(A) = A - A^t$ where A^t is the transpose of A . Compute the rank of S .

- Let V and W be finite dimensional vector spaces over a field \mathbb{F} , let V^* and W^* be the dual spaces to V and W and let $T : V \rightarrow W$ be a linear map.
 - Give the definition of V^* and show $\dim V = \dim V^*$.
 - If $S \subset V$ define the annihilator S° of S in V^* and prove it is a subspace of V^* .
 - Define the adjoint map $T^* : W^* \rightarrow V^*$.
 - Show that $\ker(T)^\circ = \text{Image } T^*$
- Suppose that A is a 3×3 real orthogonal matrix, i.e., $A^t = A^{-1}$, with determinant -1 . Prove that -1 is an eigenvalue of for A .

JANUARY 1997

- Let $M_{n \times n}$ be the vector space of all $n \times n$ real matrices.
 - Show that every $A \in M_{n \times n}$ is similar to its transpose.
 - Is there a single invertible $S \in M_{n \times n}$ so that $SAS^{-1} = A^t$ for all $A \in M_{n \times n}$?
- Let A be a 3×3 matrix over the real numbers and assume that $f(A) = 0$ where $f(x) = x^2(x-1)^2(x-2)$. Then give a complete list of the possible values of $\det(A)$.
- Show that for every polynomial $p(x) \in \mathbb{C}[x]$ of degree n there is a polynomial $q(x)$ of degree $\leq n$ so that

$$(x+1)^n f\left(\frac{x-1}{x+1}\right) = p(x).$$

HINT: Let \mathcal{P}_n be the vector space of polynomials of degree $\leq n$ and for each $f(x) \in \mathcal{P}_n$ define $(Sf)(x) := (x+1)^n f((x-1)/(x+1))$. Show that S maps $\mathcal{P}_n \rightarrow \mathcal{P}_n$ and is linear. What is the null space of S ?

AUGUST 1997

- Let V be a finite dimensional vector space and $L \in \text{Hom}(V, V)$ such that L and L^2 have the same nullity. Show that $V = \ker L \oplus \text{Im } L$.
- Let A be an $n \times n$ matrix and $n > 1$. Show that $\text{adj}(\text{adj}(A)) = \det(A^{n-2})A$.
- Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 3 & -6 & 6 \end{bmatrix}$. Compute the rational canonical form and the Jordan canonical form of A .
- Let A be an $n \times n$ real matrix such that $A^3 = A$. Show that the rank of A is greater than or equal to the trace of A .
- Let $A = [a_{ij}]$ be a real $n \times n$ matrix with positive diagonal entries such that

$$a_{ii}a_{jj} > \sum_{k \neq i} |a_{ik}| \sum_{l \neq j} |a_{il}|$$

for all i, j . Show that $\det(A) > 0$. HINT: Show first that $\det(A) \neq 0$.

JANUARY 1998

- For any nonzero scalar a , show that there are no real $n \times n$ matrices A and B such that $AB - BA = aI$.
- Let V be a vector space over the rational numbers \mathbb{Q} with $\dim V = 6$ and let T be a nonzero linear operator on V .
 - If $f(T) = 0$ for $f(x) = x^6 + 36x^4 - 6x^2 + 12$, determine the rational canonical form for T (and prove your result is correct).
 - Is T an automorphism of V ? If so describe T^{-1} ; if not describe why not.
- Suppose that A and B are diagonalizable matrices over a field \mathbb{F} . Prove that they are simultaneously diagonalizable, that is there exists an invertible matrix P such that PAP^{-1} and PBP^{-1} are both diagonal, if and only if $AB = BA$.

AUGUST 1998

1. V be a finite dimensional vector space and let W be a subspace of V . Let $\mathcal{L}(V)$ the set of linear operators on V and set $Z = \{T \in \mathcal{L}(V) : W \subseteq \ker(T)\}$. Prove that Z is a subspace of $\mathcal{L}(V)$ and compute its dimension in terms of the dimensions of V and W .
2. Let V be a finite dimensional vector space and $\mathcal{L}(V)$ the set of linear operators on V . Suppose $T \in \mathcal{L}(V)$. Suppose that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

where V_i is T invariant for each $i \in \{1, \dots, r\}$. Let $m(x)$ be the minimal polynomial of T and $m_i(x)$ the minimal polynomial of T restricted to V_i , for each $i \in \{1, \dots, r\}$. How is $m(x)$ related to the set $\{m_1(x), \dots, m_r(x)\}$.

3. Let V be a finite dimensional vector space and $\mathcal{L}(V)$ the set of linear operators on V . Let $S, T \in \mathcal{L}(V)$ so that $S+T = I$ and $\dim \text{range } S + \dim \text{range } T = \dim V$. Prove that $V = \text{range } S \oplus \text{range } T$ and that $ST = TS = 0$.
4. Let A and B be $n \times n$ matrices. Suppose that A^k and B^k have the same minimal polynomials and the same characteristic polynomials for $k = 1, 2$, and 3 . Must A and B be similar? If so prove it. If not, give a counterexample.

JANUARY 1999

1. Let V be a finite dimensional vector space and let $T : V \rightarrow V$ be a linear transformation which is not zero and is not an isomorphism. Prove there exists a linear transformation S so that $ST = 0$, but $TS \neq 0$.
2. Let T be a linear operator on the finite dimensional vector space V . Prove that if $T^2 = T$, then $V = \ker T \oplus \text{image } T$.
3. Let S and T be 5×5 nilpotent matrices with $\text{rank } S = \text{rank } T$ and $\text{rank } S^2 = \text{rank } T^2$. Are S and T necessarily similar? Prove or give a counterexample.
4. Let A and B be $n \times n$ matrices over \mathbb{C} with $AB = BA$. Prove A and B have a common eigenvector. Do A and B have a common eigenvalue.

AUGUST 1999

1. Make a list, as long as possible, of square matrices over \mathbb{C} such that
 - (a) Each matrix on the list has characteristic polynomial $(x - 2)^4(x - 3)^4$,
 - (b) Each matrix on the list has minimal polynomial $(x - 2)^2(x - 3)^2$, and,
 - (c) No matrix on the list is similar to a matrix occurring elsewhere on the list.
 Demonstrate that your list has all the desired attributes.
2. Let A and B be nilpotent matrices over \mathbb{C} .
 - (a) Prove that if $AB = BA$, then $A + B$ is nilpotent.
 - (b) Prove that $I - A$ is invertible.
3. Let V be a finite dimensional vector space. Recall that for $X \subseteq V$ the set X° is defined to be $\{f \mid f \text{ is a linear functional of } V \text{ and } f(x) = 0 \text{ for all } x \in X\}$. Let U and W be subspaces of V . Prove the following
 - (a) $(U + W)^\circ = U^\circ \cap W^\circ$.
 - (b) $U^\circ + W^\circ = (U \cap W)^\circ$.