CHANGE OF BASES AND SUMS OF POWERS OF INTEGERS

RALPH HOWARD
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, S.C. 29208, USA
HOWARD@MATH.SC.EDU

1. Some Bases of the Polynomials of Degree $\leq n$.

Let $\mathcal{P}_n$ be the vector space of real polynomials of degree $\leq n$. That is

$$\mathcal{P}_n := \{a_0 + a_1 x + \cdots + a_n x^n : a_0, \dots, a_n \in \mathbf{R}\}.$$

We have seen that $\dim \mathcal{P}_n = n + 1$ and that it has the "usual" basis

$$\mathcal{U} := \{1, x, \dots, x^n\}$$

coming from the powers of x. In different problems there different bases of $\mathcal{P}_n$ that are better adapted to the problem. For example let $a \in \mathbf{R}$ then the ordered basis

$$\mathcal{A} := \left\{1, (x-a), \frac{(x-a)^2}{2}, \frac{(x-a)^3}{3!}, \dots, \frac{(x-a)^n}{n!}\right\}$$

has the property that if $f(x) \in \mathcal{P}_n$ is expressed in this basis:

$$f(x) = \sum_{k=0}^{n} a_k \frac{(x-a)^k}{k!} \tag{1.1}$$

then the coordinate $a_0, \dots, a_n$ are given by

$$a_k = f^{(k)}(a)$$

where $f^{(k)}$ is the k-th derivative of $f(x)$ (and $f^{(0)} = f$ be definition).

Problem 1. Derive this formula for a_k by taking the k-th derivative of both sides of (1.1) and then setting $x = a$. (Note this is exactly the usual derivation of the coefficients in a Taylor series that you know and love from calculus.) □

In our terminology the coordinate vector of the vector $f(x) \in \mathcal{P}_n$ is

$$[f(x)]_{\mathcal{A}} = \begin{bmatrix} f(a) \\ f'(a) \\ f''(a) \\ \vdots \\ f^{(n-1)}(a) \\ f^{(n)}(a) \end{bmatrix}$$

which is just the list of the values of the derivatives of $f(x)$ at $x = a$. So in a context where one is working with the derivatives of polynomials at the point $x = a$ the basis $\mathcal{A}$ is the natural one to use.

2. Formulas for Sums of Powers.

We now give a basis of $\mathcal{P}_n$ were it is easy to derive "summation formulas" (the precise meaning of this will be cleared up below) and then by expessing the usual basis $\{1, x, x^2, \dots, x^n\}$ in terms of this basis we can derive formulas for sums of powers. This is a theme we will see repeatedly during the term: Make a problem easier by changing to a nicer basis. Set

$$S_0(x) := 1$$

and for $1 \leq k \leq n$ set

$$S_k(x) := x(x-1)(x-2)\cdots(x-k+1).$$

This has k factors and so has degree k. For small values of k we have

$$\begin{aligned} S_0(x) &= 1, \\ S_1(x) &= x, \\ S_2(x) &= x(x-1), \\ S_3(x) &= x(x-1)(x-2), \\ S_4(x) &= x(x-1)(x-2)(x-3). \end{aligned}$$

Let

$$\mathcal{S} := \{S_0(x), S_1(x), \dots, S_n(x)\}.$$

Then $\mathcal{S}$ is an ordered basis of $\mathcal{P}_n$. We now define the change of basis matrices between the usual basis $\mathcal{U}$ and the basis $\mathcal{S}$ by

$$S_k(x) = \sum_{i=0}^{k} a_{ki} x^i$$

and

$$x^k = \sum_{i=0}^{k} b_{ki} S_i(x). \tag{2.1}$$

So for example

$$S_3(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x = a_{33}x^3 + a_{32}x^2 + a_{31}x + a_{30}1$$

which implies

$$a_{33} = 1, \quad a_{32} = -3, \quad a_{31} = 2, \quad a_{30} = 0.$$

Likewise

$$x^3 = S_3(x) + 3S_2(x) + S_1(x) = b_{33}S_3(x) + b_{32}S_2(x) + b_{31}S_1(x) + b_{30}S_0(x)$$

yielding

$$b_{33} = 1, \quad b_{32} = 3, \quad b_{31} = 1, \quad b_{30} = 0.$$

Problem 2. Show that $AB = BA = I_{n+1}$ on $\mathcal{P}_n$. □

On $\mathcal{P}_6$ the matrices $A = [a_{ki}]$ and $B = [b_{ki}]$ are 7 by 7 matrices given explicitly by

$$A = [a_{ki}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 & 0 & 0 \\ 0 & -6 & 11 & -6 & 1 & 0 & 0 \\ 0 & 24 & -50 & 35 & -10 & 1 & 0 \\ 0 & -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}$$

and

$$B = [b_{ki}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}.$$

In matrix notation this means

$$\begin{bmatrix} S_0(x) \\ S_1(x) \\ S_2(x) \\ S_3(x) \\ S_4(x) \\ S_5(x) \\ S_6(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 & 0 & 0 \\ 0 & -6 & 11 & -6 & 1 & 0 & 0 \\ 0 & 24 & -50 & 35 & -10 & 1 & 0 \\ 0 & -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix} \begin{bmatrix} S_0(x) \\ S_1(x) \\ S_2(x) \\ S_3(x) \\ S_4(x) \\ S_5(x) \\ S_6(x) \end{bmatrix}.$$

Problem 3. Check that the the values for a_{ki} and b_{ki} are correct for $i, k \leq 4$. □

There are some obvious patterns in these matrices.

Problem 4. Show the following:

1. $a_{kk} = b_{kk} = 1$ for all $k \geq 0$.
2. $a_{k0} = b_{k0} = 0$ for $k \geq 1$.
3. The signs in the matrix A have a chess board pattern. That is $(-1)^{i+k} a_{ki} \geq 0$.
4. $a_{k\,k-1} = -b_{k\,k-1}$. (Not easy, use that $AB = I$.)
5. $b_{ki} \geq 0$ for all i, k. (Hard.)
6. $b_{k1} = 1$ for $k \geq 1$. (Hard.) □

From the point of view of summation formulas what makes the $S_k(x)$ nice is the relation:

$$S_k(x) = \frac{1}{k+1}(S_{k+1}(x+1) - S_{k+1}(x))$$

which holds for $k \geq 0$.

Problem 5. Verify this formula. □

The standard trick with telescoping series shows that

$$\begin{aligned} S_k(0) + S_k(1) + S_k(2) + \cdots + S_k(N) &= \frac{1}{k+1}(S_{k+1}(N+1) - S_{k+1}(0)) \\ &= \frac{1}{k+1} S_{k+1}(N+1). \end{aligned}$$

(At the last step we have used $S_{k+1}(0) = 0$.)

Problem 6. Verify this. □

In summation notation this is

$$\sum_{j=0}^{N} S_k(j) = \frac{1}{k+1} S_{k+1}(N+1). \tag{2.2}$$

Now we can give formulas for sums of powers of integers. Using equations (2.1) and (2.2) we have

$$\begin{aligned}\sum_{j=0}^{N} j^k &= \sum_{j=0}^{N}\sum_{i=0}^{k} b_{ki} S_i(j)\\ &= \sum_{i=0}^{k} b_{ki} \sum_{j=0}^{N} S_i(j)\\ &= \sum_{i=0}^{k} b_{ki} \frac{1}{i+1} S_{i+1}(N+1).\end{aligned}$$

For small values of k this gives

$$\begin{aligned}\sum_{j=0}^{N} j =& b_{11}\frac{1}{2}S_2(N+1) + b_{10}S_1(N+1)\\ \sum_{j=0}^{N} j^2 =& b_{22}\frac{1}{3}S_3(N+1) + b_{21}\frac{1}{2}S_1(N+1) + b_{20}S_0(N+1)\\ \sum_{j=0}^{N} j^3 =& b_{33}\frac{1}{4}S_3(N+1) + b_{32}\frac{1}{3}S_1(N+1) + b_{31}\frac{1}{2}S_0(N+1)\\ &+ b_{30}S_0(N+1).\end{aligned}$$

Problem 7. Use these to derive the familiar formulas for $\sum_{j=0}^{N} j$, $\sum_{j=0}^{N} j^2$, and $\sum_{j=0}^{N} j^3$. □