Mathematics 700 Homework
Due Wednesday, October 13

Some more on matrices of linear maps. The following shows that it is possible to choose a bases for the range and domain of a linear map that puts makes the matrix particularly simple.

Problem 1. Let V and W be finite dimensional vector spaces and in let $T\colon V \to W$ be a linear.

1. First assume that $\dim V = 5$, $\dim W = 6$ and that $\operatorname{rank} T = 3$. Then by rank plus nullity we have $\operatorname{nullity} T = 2$. Choose a basis v_4, v_5 of $\ker T$. Then these can be extended to a basis $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5\}$ of V (so the last two in the list are the basis of $\ker T$.) Let $w_1 = Tv_1, w_2 = Tv_2, w_3 = Tv_3$. Than as $\operatorname{Span}\{v_1, v_2, v_3\} \cap \ker T = \{0\}$ it follows that w_1, w_2, w_3 are linearly independent. Thus we can extend $\{w_1, w_2, w_3\}$ to a basis $\mathcal{W} = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ of W. Then show that in these bases the matrix of T is
$$[T]^{\mathcal{W}}_{\mathcal{V}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
2. Now assume that $\dim V = m$, $\dim W = n$, and that $\operatorname{rank} T = r$. Then show show that it is possible to choose bases $\mathcal{V}$ for V and $\mathcal{W}$ for W so that the matrix $[T]^{\mathcal{W}}_{\mathcal{V}}$ has r ones down the main diaginal and all other element zero.

Quotient Spaces. If there is any one idea the characterizes modern algebra it is the idea of a quotient structure. The following problems introduce the linear algebra version of this concept. You may have seen other versions as in the integers mod n, the quotient of a group by a normal subgroup, and the quotient of a ring by an ideal.

Let V be a vector space over the field $\mathbf{F}$ and W a subspace of V. Then define an equivalence relation $\sim_W$ by
$$v_1 \sim_W v_2 \quad \text{if and only if} \quad v_2 - v_1 \in W.$$

Problem 1. Show that this is an equivalence relation. (Recall a relation $\sim$ on a set V is an ***equivalence relation*** iff the three conditions (1) $x \sim x$ for all $x \in V$ (it is ***reflective***) (2) $x \sim y$ implies $y \sim x$ for all $x, y \in V$ ($\sim$ is ***symmetric***) and (3) $x \sim y$ and $y \sim z$ implies $x \sim y$ ($\sim$ is ***transitive***) hold.) $\square$

Denote by $[v]_W$ the equivalence class of $v \in V$ under the equivalence relation $\sim_W$. That is
$$[v]_W := \{u \in V : u \sim_W v\}.$$

Problem 2. Show $[v]_W = v + W$ where $v + W = \{v + w : w \in W\}$. $\square$

Let V/W be the set of all equivalence classes of $\sim_W$. That is

$$V/W := \{[v]_W : v \in V\} = \{v + W : v \in V\}.$$

The equivalence class $[v]_W = v + W$ is often called the ***coset of*** v ***in*** V/W.

Problem 3. Let $V = \mathbf{R}^2$ and let W be the subspace of points of V of points (x, y) with $y = -2x$. Then draw pictures of the coset of $(1, 1)$ in V/W and the coset of $(3, -2)$ in V/W. What is a geometric description of the coset of $v \in \mathbf{R}^2$ in V/W? $\square$

Define a sum and scalar multiplication in V/W by

$$[v_1]_W + [v_2]_W := [v_1 + v_2]_W\,, \quad c[v]_W := [cv]_W$$

where $v_1, v_2, v \in V$ and $c \in \mathbf{F}$.

Problem 4. Show this is well defined. Recall the term ***well defined*** is used in mathematics to mean "is independent of the choices made in the definition". In this particular case this means you need to show

$$[v_1]_W = [v_1']_W \quad \text{and} \quad [v_2]_W = [v_2']_W \qquad \text{implies} \qquad [v_1 + v_2]_W = [v_1' + v_2']_W$$

and

$$[v]_W = [v']_W \qquad \text{implies} \qquad [cv]_W = [cv']_W.$$

Proposition 1. *With these operations V/W is a vector space.* $\square$

Problem 5. Prove this. $\square$

Define a map $\pi\colon V \to V/W$ by $\pi(v) = [v]_W$. (Or in slightly different notation $\pi v = v + W$.) This is the ***natural projection*** or ***canonical projection*** of V onto the quotient space V/W.

Problem 6. The natural projection $\pi\colon V \to V/W$ is a linear map and $\ker(\pi) = W$. $\square$

Problem 7. If V is finite dimensional then what is the dimension of V/W in terms of $\dim V$ and $\dim W$? Prove your answer is correct. (HINT: Rank plus nullity.) $\square$

Problem 8. In the example of Problem 3 draw some pictures of cosets $v_1 + W$ and $v_2 + W$ what their sum $(v_1 + W) + (v_2 + W)$ and the linear combination $2(v_1 + W) - 3(v_2 + W)$ for a few choices of v_1 and v_2. $\square$

We now give a very basic result. In the context of groups this is called ***the first homomorphism theorem***.

Theorem 1. *Let V and U be vector spaces and let $T\colon V \to U$ be a surjective (that is onto) linear map. Then the vector space $V/\ker T$ is isomorphic to U. (Written as $U \approx V/\ker T$.)* $\square$

We are not assuming that V and U are finite dimensional so it is not enough to just compute dimensions.

Problem 9. Prove this along the following lines: First to simplify notation set $W := \ker T$ so that

$$W = \{v \in V : Tv = 0\}.$$

Then we wish to show that $V/W \approx U$. Define a map $\widetilde{T} \colon V/W \to U$ by

$$\widetilde{T}[v]_W = Tv.$$

(a) Show that $\widetilde{T}$ is well defined. (That is if $[v_1]_W = [v_2]_W$ then $Tv_1 = Tv_2$.)
(b) Show that $\widetilde{T}$ is linear.
(c) Show that $\widetilde{T}$ is onto (i.e. surjective). (Hint: Use that T is onto so that every $u \in U$ is of the form $u = Tv$ for some $v \in V$.)
(d) Show that $\widetilde{T}$ is one to one (i.e. injective). (Hint: You wish to show that if $\widetilde{T}[v]_W = 0$ then $[v]_W = [0]_W$. But $[v]_W = [0]_W$ if and only if $v \in W = \ker T$.) $\square$