Mathematics 700 Homework
due Wednesday, October 6

The following are problems on finding the matrices of linear maps. There are examples in Chapter 10 of the text that are relevant to these problems.

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by
$$T(x, y) := (x - 2y, -4x + y, 7x - 11y).$$
Then find the matrix of T with respect to the standard bases of each of $\mathbb{R}^2$ and $\mathbb{R}^3$. (Recall that the standard basis of $\mathbb{R}^n$ is the basis $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $e_3 = (0, 0, 1, \ldots, 0) \ldots$.)
2. With T as in the last problem find the matrix of T with respect to the bases $\mathcal{V} = \{(1, 2), (3, 2)\}$ of $\mathbb{R}^2$ and $\mathcal{W} := \{(1, 1, 1), (0, 2, 1), (0, 0, 3)\}$ of $\mathbb{R}^3$.
3. Letting $\mathcal{P}_3$ be the real polynomials of degree ≤ 3 and using the standard basis $\mathcal{V} := \{1, x, x^2, x^3\}$ of $\mathcal{P}_3$ find the matrices, the rank and the nullity of the following linear maps
(a) $(Tp)(x) = p(x - 2)$,
(b) $(Cp)(x) = (x + 1)^3 p\left(\dfrac{x - 1}{x + 1}\right)$,
(c) $Ap = p + p' + p'' + p''' + p''''$,
(d) $(Pp)(x) = e^{-x} \displaystyle\int_{-\infty}^{x} p(t)e^t\, dt$.
(e) $(Bp)(x) = \displaystyle\int_0^x p'(t)\, dt$
(f) $(Vp)(x) = \dfrac{d}{dx} \displaystyle\int_0^x p(t)\, dt$
4. Let $M_{2\times 2}$ be the vector space of 2×2 matrices over the real numbers. Let
$$A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$
Use for $M_{2\times 2}$ the the ordered basis basis $\mathcal{V} := \{E_{11}, E_{12}, E_{21}, E_{22}$ where
$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$
Then find the matrics of the following linear maps from $M_{2\times 2}$ to itself.
(a) $LX = AX$,
(b) $RX = XA$,
(c) $CX = AX - XA$,

(d) $TX = X^t$ (the transpose of X),
(e) $SX = \frac{1}{2}(X + X^t)$ and find the rank and nullity of this map, and
(f) $GX = \frac{1}{2}(X - X^t)$ and find the rank and nullity of this map.

5. The set of complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ is a two dimensional vector space over the real numbers $\mathbb{R}$. Using the basis $\mathcal{B} = \{1, i\}$ for this real vector space find the matrices of the following linear maps
(a) $Jz = iz$,
(b) $Cz = \overline{z}$ (where $\overline{z}$ is the complex conjugate of z),
(c) $Tz = (2 + 3i)z$,
(d) $Mz = (a + bi)z$, and
(e) $Rz = \frac{1}{2}(z + \overline{z})$.