Mathematics 700 Homework
Due Monday, November 22

This is to catch some lose ends and review what we have done to date. Most of these are taken off of old admission to candidacy exams.

Problem 1. Let n be a positive integer. Define

$$
\begin{aligned}
G =& \{A : \text{ } A \text{ is an } n \times n \text{ matrix with only integer entries and } \det A \in \{-1, +1\}\}, \\
H =& \{A : \text{ } A \text{ is an invertible } n \times n \text{ matrix and both } A \text{ and } A^{-1} \text{ have only} \\
& \quad \text{integer entries}\}.
\end{aligned}
$$

Prove $G = H$. (August 1984)

Problem 2. Let V be the vector space over $\mathbf{R}$ of all $n \times n$ matrices with entries from $\mathbf{R}$.

1. Prove that $\{I, A, A^2, \ldots, A^n\}$ is linearly dependent for all $A \in V$.
2. Let $A \in V$. Prove that A is invertible if and only if I belongs to the span of $\{A, A^2, \ldots, A^n\}$. (August 1984)

Problem 3. Let $T \in L(V, V)$, where V is a finite dimensional vector space. (Here $L(V, V)$ is the set of linear operators on V. or a linear operator S denote by $\mathcal{N}(S)$ the null space and by $\mathcal{R}(S)$ the range of S.)

1. Prove there is a least natural number k such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1}) = \mathcal{N}(T^{k+2}) = \cdots$ Use this k in the rest of this problem.
2. Prove that $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1}) = \mathcal{R}(T^{k+2}) = \cdots$
3. Prove that $\mathcal{N}(T^k) \cap \mathcal{R}(T^k) = \{0\}$.
4. Prove that for each $\alpha \in V$ there is exactly one vector in $\alpha_1 \in \mathcal{N}(T^k)$ and exactly one vector $\alpha_2 \in \mathcal{R}(T^k)$ such that $\alpha = \alpha_1 + \alpha_2$. (January 1985 and January 1992)

Problem 4. Prove that if A and B are $n \times n$ matrices from $\mathbf{C}$ and $AB = BA$, then A and B have a common eigenvector. (August 1985) Hint: Let α be an eigenvalue of A let $V_\alpha = \{v \in \mathbf{C}^n : Av = \alpha v\}$ be the corresponding eigenspace. Show that V_α is invariant under B and consider the restriction $B\big|_{V_\alpha}$ of B to V_α.

Problem 5. Compute the minimal and characteristic polynomials of the following matrix. Is it diagonalizable?

$$
\begin{bmatrix}
5 & -2 & 0 & 0 \\
6 & -2 & 0 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 1 & -1
\end{bmatrix}
$$

(August 1987)

Problem 6. Find the invariant factors and Smith normal form of

$$A \begin{bmatrix} x & 0 & 0 & 3 \\ -1 & x^2 & 0 & 4 \\ 0 & -1 & x^3 & 5 \\ 0 & 0 & -1 & x^4+6 \end{bmatrix}.$$

HINT: You can find 1×1, 2×2 and 3×3 subdeterimnats that are unit. You can also do row operations to make it upper triangular.

Problem 7. Here is a determinant formula that every mathematician seems to know and love. Let

$$V(x_1, x_2, \ldots, x_n) = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_n^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

Show

$$\det V(x_1, x_2, \ldots, x_n) = \prod_{i<j} (x_j - x_i)$$

HINT: Use induction. Multiply each column by x_1 and substract it from the next column its the right (this will not change the value of the determinant). Then you should find that

$$\det V(x_1, x_2, \ldots, x_n) = (x_n - x_1)(x_{n-1} - x_1) \cdots (x_2 - x_1) \det V(x_2, x_3, \ldots, x_n)$$