Mathematics 552 Test #3 Name: ______________________

Show your work! Answers that do not have a justification will receive no credit. Also put your answers in some standard "human" form such as $x + iy$ or polar form, and not some specialized form used by your calculator.

(1) (20 points)

(a) Define entire function.

(b) State Liouville's Theorem.

(c) State the fundamental theorem of algebra.

(d) Define what it means for z_0 to be an isolated singularity of $f(z)$.

(e) State the interior maximum modulus principle.

(f) If z_0 is an isolated singularity of $f(z)$, then define the residue of $f(z)$ at z_0.

(g) State the Residue Theorem.

(2) (20 points) Let $f(z)$ have an isolated singularity at z_0.

(a) State theorem on the existence of Laurent expansions of $f(z)$ about z_0.

(b) Define what if means for z_0 to be a removable singularity of $f(z)$.

(c) Define what it means for z_0 to be a pole of $f(z)$.

(d) Define what it means for z_0 to be an essential singularity of $f(z)$.

(e) State the theorem that characterizes when z_0 is a removable singularity of $f(z)$.

(f) State the theorem that characterizes when z_0 is a pole of $f(z)$.

(g) State the structure theorem for poles of order k.

(3) (15 points)

(a) State the mean value property theorem for analytic functions.

(b) Derive mean value property theorem for analytic functions from the Cauchy integral formula.

(4) (15 points)

(a) Define what it means for $u(z) = u(x, y)$ to be harmonic in a domain D.

(b) Show that if u is harmonic then $f(z) = u_x - iu_y$ is analytic in D. HINT: Cauchy-Riemannian equations.

(5) (10 points) Show that if $f(z)$ has a simple pole at z_0, then $g(z) = (z - z_0)f(z)$ has removable singularity at z_0. HINT: Structure theorem for poles of order k (in this case $k = 1$).

(6) (5 points) Show that if $f(z)$ is an entire function such that $|f(z) - 3i| \geq 2$ for all z, then $f(z)$ is constant.

(7) (10 points) Let D be the unit disk $\{z : |z| < 1\}$ and let u be a harmonic function on D such that $u(z) = x^2 - y^2$ for all $z \in \partial D$ where $z = x + iy$. Show that $u(z) = x^2 - y^2$ for all $z \in D$. Hint: Is $v(z) = x^2 - y^2$ harmonic?