Mathematics 552, Review for Test 3

Reminder: The test is Monday, April 17.

The best thing to do is to go over the quizzes and homework. I will look at these while making up the test. You can find these (hopefully with most of the typos corrected) at

`http://www.math.sc.edu/~howard/Classes/552d/`

Also anything that is proven on this sheet is fair game for the test.

The last test brought us up to the Cauchy integral formula. We when showed that his implied such a formula for the derivatives of an analytic.

Theorem (Existance of higher derivatives)**.** *If $f(z)$, is analytic in a domain D, then its derivative $f'(z)$ is also analytic in D. This all the higher derivatives $f'(z), f''(z), \dots, f^{(n)}(z), \dots$ exist and are analytic in D.* □

Theorem (Cauchy formula for higher derivatives)**.** *Let D be a bounded domain with nice boundary and $f(z)$ a function that is analytic in D and continuous in $D \cup \partial D$. Then the n-th derivative of $f(z)$ is given by*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

□

Definition (Entire function)**.** A function is ***entire*** iff it is analytic in all of $\mathbf{C}$. □

We used the formula

$$f'(a) = \frac{1}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^2} \, dz \tag{1}$$

to show that if $f(z)$ was a bounded entire function that $f'(z) \equiv 0$. (This was done by taking a limit as $R \to \infty$.) Here is one way to see this. In equation (1) we parmeterize the circle $|z - a| = R$ by $z = a + Re^{it}$ with $0 \le t \le 2\pi$. Then $z - a = Re^{it}$ and $dz = iRe^{it} \, dt$. Thus the formula (1) for $f'(a)$ becomes

$$f'(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + Re^{it})}{(Re^{it})^2} iRe^{it} \, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(a + Re^{it})}{R} e^{-it} \, dt$$

If $f(z)$ is entire and bounded then $|f(z)| \le M$ for some constant M. Thus

$$\begin{aligned}
|f'(a)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{f(a + Re^{it})}{R} e^{-it} \, dt \right| && \text{(by equation (1))} \\
&\le \int_0^{2\pi} \frac{|f(a + Re^{it})|}{R} |e^{-it}| \, dt && \text{(putting absolute values inside the integral)} \\
&= \int_0^{2\pi} \frac{|f(a + Re^{it})|}{R} | \, dt && \text{(as } |e^{it}| = 1\text{)} \\
&\le \int_0^{2\pi} \frac{M}{R} \, dt && \text{(as } |f(a + Re^{it})| \le M\text{)} \\
&= \frac{2\pi M}{R}.
\end{aligned}$$

Therefore

$$0 \le |f'(a)| \le \lim_{R\to\infty} \frac{2\pi M}{R} = 0,$$

which shows that $f'(a) = 0$. This holds for all $a \in \mathbf{C}$ and thus $f'(z) \equiv 0$. But an analytic function with zero derivative is constant. Thus we have derived:

Theorem (Liouville's Theorem)**.** *A bounded entire function is constant.* □

A main application of this is:

Corollary (Fundamental Theorem of Algebra)**.** *If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a non-constant polynomial. This $p(z)$ has a complex root. That is there is a complex number r such that $p(r) = 0$.* □

Another nice application is

Corollary (Casorati-Weierstrass Theorem)**.** *If $f(z)$ is non-constant entire function, then the image of $f(z)$ meets each disk $D(a, r) = \{z : |z - a| < r\}$. That is for each such disk there is a $z \in \mathbf{C}$ with $f(z) \in D(a, r)$.*

Proof. Assume, toward a contradiction, that there is a non-constant entire function $f(z)$ that does not have any values in some disk $D(a, r)$. This means that $|f(z)-a| \ge r$ for all $z \in \mathbf{C}$. Then $h(z) = \dfrac{1}{f(z) - a}$ is an entire function (as $f(z) \ne a$ for all z) and $|f(z) - a| \ge r$ implies

$$|g(z)| = \frac{1}{|f(z) - a|} \le \frac{1}{r}.$$

Thus $h(z)$ is a bounded entire function and so, by Liouville's Theorem, $h(z)$ is constant. But then $f(z) = a + \frac{1}{h(z)}$ is constant, contradicting our assumption that $f(z)$ is non-constant. □

Our next application of the Cauchy integral formula was to apply it to the disk $D = \{z : |z - a| < r\}$ and use it to compute the value $f(a)$ at the center of the disk. This gives

$$f(a) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{(z - a)}\, dz.$$

But in this case we can parameterize $|z - a| = r$ by $z = a + re^{it}$ with $0 \le t \le 2\pi$. Then $dz = ire^{it}\, dt$ and so

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i}t)}{re^{it}}\, ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt.$$

That is the average value of $f(z)$ over the circle $|z - a| = r$ is value of f at the center of the circle. We give this fact a name.

Theorem (Mean Value Property of Analytic Functions)**.** *Let $f(z)$ be analytic in a domain D. Then for every disk $|z - a| \le r$ contained in this domain*

(2) $$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt.$$

□

Note that the proof of the mean value property is quite easy and you will be expected to be able to reproduce it.

The application we gave of the mean value property was the maximum modulus principle. Here we give two versions.

Theorem (Interior maximum modulus principle)**.** *If $f(z)$ is analytic in a domain D and $|f(z)|$ has a maximum at some point of D, then $f(z)$ is constant.* □

Theorem (Boundary maximum modulus principle)**.** *Let $f(z)$ be analytic in a bounded domain D and continuous in $D \cup \partial D$. Then the maximum of $|f(z)|$ occurs on the boundary of D. That is*

$$\max_{z \in D \cup \partial D} |f(z)| = \max_{z \in \partial D} |f(z)|.$$

□

One application of the maximum modulus principle is Schwartz's lemma.

Theorem (Schwartz's Lemma)**.** *Let $D = \{z : |z| < 1\}$ be the unit disk, and let $f(z)$ be analytic in D and continuous in $D \cup \partial D$. Assume that $f(0) = 0$ and $|f(z)| \leq 1$. Then*

$$|f(z)| \leq |z|$$

in D. If equality holds for some $z_1 \neq 0$, then $f(z) = az$ for some constant a with $|a| = 1$. □

Definition (Harmonic function)**.** If $u(x, y)$ is a real valued function defined on a domain D then u is ***harmonic*** iff

$$\Delta u = u_{xx} + u_{yy} = 0.$$

(The operator Δ is defined by $\Delta h = h_{xx} + h_{yy}$ and we have shown this is a linear operator.) □

Remark. The operator Δ is defined by $\Delta h = h_{xx} + h_{yy}$ and we have shown this is a linear operator. We have shown that it is linear. That is $\Delta(h_1 + h + 2) = \Delta h_1 + \Delta h_2$ and $\Delta(ch) = c\Delta h$ where c is a constant. Thus if u_1 and u_2 are harmonic, that is $\Delta u_1 = 0$ and $\Delta u_2 = 0$. Then $\Delta(u_1 + u + 2) = \Delta u_1 + \Delta u_2 = 0 + 0 = 0$ and $\Delta(u_1 - u + 2) = \Delta u_1 - \Delta u_2 = 0 - 0 = 0$. Therefore $u_1 + u_2$ and $u_1 - u_2$ are also harmonic. □

Harmonic functions are closely related to analytic functions. We have seen (it was a problem in the first test) that the real part of an analytic function is always harmonic. There is a partial converse:

Theorem (Harmonic functions are real parts of analytic functions)**.** *Let D be a simply connected domain. Then the following are equivalent.*

(1) *$u(z)$ is a harmonic function on D.*
(2) *There is an analytic function $f(z)$ in D such that $u(z) = \mathrm{Re}(f(z))$.* □

Remark. This is false if the domain is not simply connected. For example $u(x, y) = \ln(x^2 + y^2)$ is harmonic in $D = \{z : z \neq 0\}$ but is not the real part of any analytic function defined on D. (It is the real part of $f(z) = 2\log(z)$, but this is not single valued in D.) □

Taking the real part of the main value property of analytic functions, that is (2) we get a mean value property for harmonic functions.

Theorem (Mean value property of harmonic functions)**.** *Let $u(z)$ be analytic in a domain D. Then for every disk $|z-a| \le r$ contained in this domain*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})\, dt.$$

□

This implies a maximum principle. Again we give two versions.

Theorem (Interior maximum and minimum principle for harmonic functions)**.** *If $u(z)$ is harmonic in a domain D and $u(z)$ has a maximum (or a minimum) at some point of D, then $u(z)$ is constant.* □

Theorem (Boundary maximum and minimum principle for harmonic functions)**.** *Let $u(z)$ be harmonic in a bounded domain D and continuous in $D \cup \partial D$. Then the maximum and minimum of $u(z)$ occur on the boundary of D. That is*

$$\max_{z\in D\cup\partial D} u(z) = \max_{z\in\partial D} u(z), \qquad \textit{and} \qquad \min_{z\in D\cup\partial D} u(z) = \min_{z\in\partial D} u(z).$$

□

Remark. Although these principles work for both maximums and minimums, to is traditional just to refer to the "Interior maximum principle" and Boundary maximum principles". This saves space and breath. □

Here is an application of this.

Theorem (Harmonic Functions that are zero on the boundary)**.** *Let D be a bounded domain and let u be a function that is harmonic in D and continuous on $D \cup \partial D$. If $u = 0$ on ∂D, then $u \equiv 0$ in all of D.*

Proof. We use the boundary maximum principle for harmonic functions. For any $a \in D$ we have

$$\begin{aligned}
0 &= \min_{z\in\partial D} u(z) && \text{(As } u = 0 \text{ on } \partial D\text{)}\\
&= \min_{z\in D} u(z) && \text{(by the maximum principle)}\\
&\le u(a) && \text{(by definition of the minimum)}\\
&\le \max_{z\in D} u(z) && \text{(by definition of the maximum}\\
&= \max_{z\in\partial D} u(z) && \text{(by the maximum principle)}\\
&= 0 && \text{(As } u = 0 \text{ on } \partial D\text{)}
\end{aligned}$$

Thus for all $a \in D$ we have $0 \le u(a) \le 0$. That is $u(a) = 0$ for all $a \in D$ and therefore $u \equiv 0$ in D. □

This implies

Theorem (Harmonic functions that are equal on the boundary)**.** *Let D be a bounded domain and let u_1 and u_2 be functions that are harmonic in D and continuous on $D \cup \partial D$. If $u_1 = u_2$ on ∂D, then $u_1 \equiv u_2$ in all of D.*

Proof. Let $u = u_1 - u_2$. This is harmonic and if $z \in \partial D$, then $u_1(z) = u_2(z)$ as u_1 and u_2 are equal on the boundary. Therefore $u(z) = u_1(z) - u_2(z) = 0$ for $z \in \partial D$. By then u is a harmonic function that vanishes on ∂D. So by the last theorem this implies that $u = u_1 - u_2 \equiv 0$ in D. Thus $u_1 \equiv u_2$ in D. $\square$

Getting back to more direct applications of the Cauchy integral formula, we used it and the expansion

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

(which converges whenever $|z - z_0| < |\zeta - z_0|$) to get that if D is a bounded domain with nice boundary and $f(z)$ is analytic in D and continuous on $D \cup \partial D$, then for all $z_0 \in D$ and z with $|z - z_0| < \operatorname{dist}(z_0, \partial D)$.

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)}\, d\zeta \\
&= \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left(\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \right) d\zeta \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \right) (z - z_0)^n \\
&= \sum_{n=0}^{\infty} a_n (z - z_0)^n
\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta = \frac{f^{(n)}(z_0)}{n!}. \tag{3}$$

Summarizing this calculation we have

Theorem (Existance of power series expansions)**.** *Let $f(z)$ be a function analytic in a domain D. Let $z_0 \in D$. Then $f(z)$ has a convergent power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with a_n given by equation (3)*. The radius of convergence is $\geq \operatorname{dist}(z_0, \partial D)$.* $\square$

You can expect problem on finding the radius of convergence of functions.

Corollary (A function is zero if all its derivatives at a point vanish)**.** *Let $f(z)$ be analytic in a domain. If there is a point $z_0 \in D$ such that $f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(n)}(z_0) = \cdots = 0$, then $f(z) = 0$ for all $z \in D$.*

Corollary (Order of a zero). *Let $f(z)$ be an analytic function that is not identically zero, then for each $z_0 \in D$ there is a non-negative integer k and a analytic function $h(z)$ in D such that*

$$f(z) = (z - z_0)^k h(z) \qquad \text{and} \qquad h(z_0) \neq 0.$$

The number k is the ***order of the zero*** *of $f(z)$ at z_0.* □

Remark. The order of the zero of $f(z)$ can also be defined as the smallest non-negative integer k such that $f^{(k)}(z_0) \neq 0$, but $f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$. You can use either definition.

Definition (Isolated singularity). The function $f(z)$ has an ***isolated singularity*** at z_0 iff for some $R > 0$ we have that $f(z)$ is analytic in the punctured disk $A = \{z : 0 < |z - z_0| < R\}$. □

Theorem (Existence of a Laurent expansion about an isolated singularity). *If $f(z)$ has an isolated singularity at z_0 then for some $R > 0$ there is a convergent* ***Laurent expansion***

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

which holds for $0 < |z - z_0| < R$. □

Definition (Classification of singularities). If $f(z)$ has an isolated singularity at z_0 with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

then the singularity is

(1) ***removable*** iff $a_n = 0$ for all $n \leq -1$. In this case $f(z)$ extends to an analytic function on the disk $\{z : |z - z_0| < R\}$ with power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.

(2) ***a pole*** iff there is a $k \leq -1$ so that $a_n = 0$ for $n < k$ but $a_k \neq 0$. In this case z_0 is a ***pole of order*** $-k$. In the case of $k = -1$ we also call z_0 a ***simple pole***.

Examples: The $f(z) = \dfrac{1}{z}$ has a simple pole (i.e. A pole of order one) at $z_0 = 0$, the function $f(z) = \dfrac{1}{z^k}$ has a pole of order k at $z_0 = 0$, and if $h(z_0) \neq 0$ then $f(z) = \dfrac{h(z)}{(z - z_0)^k}$ has a pole of order k at z_0.

(3) ***essential singularity*** iff there are infinitely many $n \leq -1$ with $a_n \neq 0$. □

Theorem (Characterization of removable singularities). *Let $f(z)$ have an isolated singularity at z_0. Then the following are equivalent.*

(1) *$f(z)$ has a removable singularity at z_0.*

(2) *$f(z)$ is bounded near z_0. That is there is an $R > 0$ and a constant $M > 0$ such that $|f(z)| \leq M$ for $0 < |z - z_0| < R$.* □

Theorem (Characterization of poles.)**.** *Let $f(z)$ have an isolated singularity at z_0. Then the following are equivalent.*

(1) *$f(z)$ has a pole at z_0.*
(2) $\lim_{z \to z_0} |f(z)| = \infty$. □

Theorem (Structure of poles of order k)**.** *Let $f(z)$ have an isolated singularity at z_0. Then the following are equivalent.*

(1) *z_0 is a pole of $f(z)$ of order k.*
(2) *There is a analytic function $h(z)$ defined in a neighborhood of z_0 with*

$$f(z) = \frac{h(z)}{(z - z_0)^k}, \qquad \textit{and} \qquad h(z_0) \neq 0.$$

□

Definition (Definition of residue)**.** If $f(z)$ has an isolated singularity at z_0 and the Laurent expansion of $f(z)$ about z_0 is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

then the ***residue of*** $f(z)$ ***at*** z_0 is

$$\operatorname{Res}(f, z_0) = a_{-1}.$$

That is the residue of $f(z)$ at z_0 is the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion of $f(z)$ at z_0. □

Theorem (Resdue Theorem)**.** *Let D be a bounded domain with nice boundary and $f(z)$ continuous on $D \cup \partial D$ and analytic function in D except at a finite number of points $z_1, \ldots, z_m$ (these points are isolated singularities of $f(z)$. Then*

$$\int_{\partial D} f(z)\, dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}(f, z_k).$$

That is $\int_{\partial D} f(z)\, dz$ is $2\pi i$ times the sum of the residues of $f(z)$ in D. □