Mathematics 552, Review for Test 1

Reminder: The test is Wednesday, February 15.

The best thing to do is to go over the quizzes and homework. I will look at these while making up the test.

(1) Basics about the complex numbers.
  (a) The definition of complex numbers as $x+iy$ where $x, y \in \mathbf{R}$ and $i^2 = -1$.
  (b) Basic arithmetic as done on the first couple of homeworks.
  (c) Be able so solve simple equations such as $z^2 + 4iz + 3 = 0$, along with linear equations in one and two variables.
  (d) The ***real part*** and ***complex part*** of a complex number. The ***complex conjugate*** $\overline{x+iy} = x - iy$ and its basic properties. Also the ***norm*** or ***modulus*** of a complex numbers $|z| = \sqrt{|z|^2}$. This includes the triangle inequality $|z+w| \le |z| + |w|$.
  (e) The ***polar form*** of a complex number $z = re^{i\theta}$. Be able to use this to find roots and powers of complex numbers.
  (f) Be able to identify geometric type (line, circle, etc.) from equations such as $|z-a| = r$, $|z-\alpha| = 5|z-\beta|$.

(2) Basics about mappings and sets.
  (a) Finding images of sets under elementary maps such as ***linear fractional transformations*** (also called ***Möbius transformations***). Some practice here would be the most recent homework, and the text pages 32–33, 1, 3, 8.
  (b) The basics about ***open sets***, ***closed sets***, ***boundaries***, ***connected sets***, and ***domains***. You will not be ask the definitions directly, but you might have to be able to identify such objects.

(3) Analytic functions.
  (a) The definition of an ***analytic function*** as a function that is complex differentiable. That is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}.$$

  (b) A function $f(z) = u(x,y) + iv(x,y)$ on an open set is analytic if and only if it satisfies the ***Cauchy-Riemann*** equations.

$$u_x = v_y, \qquad u_y = -v_x.$$

  If $f(z)$ is analytic then its derivative is give in terms of the partial derivatives of u and v by the formulas

$$f'(z) = u_x + iv_y = v_y - iu_y.$$

  Be able to derive the Cauchy-Riemann equations.
  (c) Basic differential calculus with analytic functions. That is be able to take derivatives.

(4) The basic transcendental functions.

(a) The functions e^z, $\cos(z)$, and $\sin(z)$ we extended from the real numbers to the complex numbers by use of their series expansions

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!},$$

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}.$$

These series were used to establish ***Euler's formulas***

$$e^{iz} = \cos(z) + i\sin(z),$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

These formula can be used to derive all the basic properties of $\cos(z)$ and $\sin(z)$ from those of e^z.

(b) Be able to solve equations such as $e^z = a$, $\cos(z) = 3$, $\sin(z) = 5$, etc.

(c) You should be able to find the real and imaginary parts of $\sin(z)$, $\cos(z)$ in terms of $\cosh(z) = \dfrac{e^z + e^{-z}}{2}$, and $\sinh(z) = \dfrac{e^z - e^{-z}}{2}$.

(d) For a general function given by a series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with radius of convergence R we have that this is analytic in $\{|z| < R\}$ and that the coefficients a_k are given by $a_k = \dfrac{f^{(k)}(0)}{k!}$. This can be used to find the series expansions of functions such as $f(z) = (1+z)^\alpha$.

(5) Various and sundry ***Surprise mystery questions***.