Mathematics 552 Homework due Friday, February 10, 2006

The most basic functions in analysis other than polynomials are the functions e^z, $\cos(z)$, $\sin(z)$. We consider e^z the most basic of these and have defined $\cos(z)$ and $\sin(z)$ by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

We also know that

$$\frac{d}{dz} e^{az} = a e^{az}, \qquad e^{a+b} = e^a e^b. \tag{1}$$

In this problems you will show this the basic properties of the trigonometric functions.

(1) Use (1) to show

$$\frac{d}{dz}\cos(z) = -\sin(z), \qquad \frac{d}{dz}\sin(z) = \cos(z).$$

(2) Use (1) to show

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b), \quad \sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b). \tag{2}$$

(3) Use (1) to show

$$\cos^2(z) + \sin^2(z) = 1.$$

It is convenient to introduce the hyperbolic functions

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}.$$

(4) Show that

$$\cosh^2(z) - \sinh^2(z) = 1.$$

(5) Show that

$$\frac{d}{dz}\cosh(z) = \sinh(z), \qquad \frac{d}{dz}\sinh(z) = \cosh(z).$$

(6) Show that

$$\sin(iz) = i\sinh(z), \qquad \cos(iz) = \cosh(z). \tag{3}$$

(7) Let $z = x + iy$ as usual. Then use the equations (2) and (3) to expand $\cos(x+iy)$ and $\sin(x+iy)$ to find the real and imaginary parts of of $\cos(z)$ and $\sin(z)$. That is write each of $\cos(z)$ and $\sin(z)$ in the form $u(x,y) + iv(x,y)$ where $u(x,y)$ and $v(x,y)$ are real valued.

(8) Use your solution to the last problem to find formulas for $|\cos(z)|^2$ and $|\sin(z)|^2$.

(9) Here is a fun, but more challenging, problem. Find a formula for the sum

$$\sum_{k=0}^{n} \cos(kz) = 1 + \cos(z) + \cos(2z) + \cdots + \cos(nz).$$

HINT: One method is to use (1) and the sum formula for a finite geometric series.