Mathematics 552 Homework due Wednesday, February 8, 2006

Here are some fun and games with series. First let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1}$$

and assume that this series has a positive radius of convergence R. Then we have seen that $f(z)$ is analytic in the disk $|z| < R$ and that its derivative is given by

$$f'(z) = \sum_{k=0}^{\infty} k a_k z^{k-1} = \sum_{k=1}^{\infty} k a_k z^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k \tag{2}$$

and that this series also has radius of convergence R. Therefore $f'(z)$ is also analytic in the disk $|z| < R$ and so we can take its derivative to get

$$f''(z) = \sum_{k=0}^{\infty} k(k-1) a_k z^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} z^k. \tag{3}$$

Problem 1. Show, for example by using mathematical induction, that we can continue in this manner and that the n-th derivative $f^{(n)}(z)$ of $f(z)$ exists in the disk $|z| < R$ and is given by the series

$$\begin{aligned} f^{(n)}(z) &= \sum_{k=0}^{\infty} k(k-1)(k-2)\dots(k-(n-1)) a_k z^{k-n} \\ &= \sum_{k=n}^{\infty} k(k-1)(k-2)\dots(k-(n-1)) a_k z^{k-n} \\ &= \sum_{k=1}^{\infty} (k+n)(k+n-1)(k+n-2)\dots(k+1) a_{k-n} z^k. \end{aligned} \tag{4}$$

Problem 2. If $f(z)$ is given by the series (1) we can let $z = 0$ to find that

$$f(0) = a_0.$$

Therefore $a_0 = f(0)$. Letting $z = 0$ in the formula (2) for $f'(z)$ gives

$$f'(0) = a_1.$$

Which gives $a_1 = f'(0)$. Letting $z = 0$ in the formula (3) for $f''(z)$ gives

$$f''(0) = 2a_2.$$

Giving $a_2 = \frac{f''(0)}{2}$. Now let $z = 0$ in (4) to find a formula for a_n in terms of $f^{(n)}(0)$. Remark: If you want to check your formula, you can find the it in the powers series section of your calculus book, or in our text.

Problem 3. Use your solution to problem 2 to find series for the following

(a) $(1+z)^{-2}$. That is in the expansion

$$(1+z)^{-2} = \sum_{k=0}^{\infty} a_k z^k$$

find a formula for a_k.

(b) It is known that $\sqrt{1+z} = (1+z)^{1/2}$ is analytic in the disk $|z| < 1$. Find the series expansion for $\sqrt{1+z}$.

(c) More generally let α be a complex number. We will see later that we can define $f(z) = (1+z)^\alpha$ in such a way that it has a power series $(1+z)^\alpha = \sum_{k=0}^\infty a_k z^k$ that converges for $|z| < 1$. Find coefficients a_k, in this expansion. REMARK: If you do this problem first, then you get the solutions to (a) and (b) let letting $\alpha = -2$ and $\alpha = 1/2$ respectively.

Problem 4. Use multiplication and long division of series to find the first four (that is up to degree three terms) of the series

(a) $\dfrac{\sin(z)}{z}$.

(b) $\tan(z) = \dfrac{\sin(z)}{\cos(z)}$.

(c) $\dfrac{e^{2z}-1}{z+z^2}$

(d) $(1+z)e^{2z}$

(e) $\dfrac{e^{3z}}{1-2z}$.

Some extra credit.

Assume that the series

$$f(z) = \sum_{k=0}^\infty a_k z^k, \qquad g(z) = \sum_{j=0}^\infty b_j z^j$$

converge in the disk $|z| < R$. Let $p(z)$ be the series obtained by multiplying this together formally. That is

$$p(z) = f(z)g(z) = \left(\sum_{k=0}^\infty a_k z^k\right)\left(\sum_{j=0}^\infty b_j z^j\right) = \sum_{n=0}^\infty \left(\sum_{k=0}^n a_k b_{n-k}\right) z^k = \sum_{n=0}^\infty c_n z^n$$

where

$$c_n = \sum_{k=0}^\infty a_k b_{n-k}.$$

We would like to show that the series for $p(z)$ also converges for $|z| < R$. So let $|z| < R$ and choose an r with $|z| < r < R$. Then the series for $f(r)$ and $g(r)$ both converge.

EC 1. Explain why this implies there is a constant $M > 0$ such that $|a_k r^k|, |b_k r^k| \le M$. This implies that

$$|a_k|, |b_k| \le \frac{M}{r^k}.$$

EC 2. Use your solution to EC 1 to show that

$$|c_n| \le (n+1)\frac{M^2}{r^n},$$

EC 3. Thus if $\rho = \dfrac{|z|}{r}$, which satisfies $\rho < 1$ as $|z| < r$, then use EC 2 to show that

$$|p(z)| \le \sum_{n=0}^\infty |c_n z^n| \le \sum_{n=0}^\infty (n+1)M^2\rho^n = M^2 \sum_{n=0}^\infty (n+1)\rho^n$$

and that the series $\sum_{n=0}^\infty (n+1)\rho^n$ converges. This shows that the series for $p(z)$ is absolutely convergent and completes the proof.