Mathematics 552 Homework due Friday, January 27, 2006

(1) Problem 3, page 33 of the text.
(2) Problem 1, page 38 of the text.
(3) Problem 2, page 38 of the text.
(4) Problem 3, page 38 of the text. Hint: Use Cauchy-Riemann equations.
(5) Problem 5, page 39 of the text.

Quiz on Friday: Be able to derive the Cauchy-Riemann equations. Here is the derivation. Assume the $f(z) = u(x, y) + iv(x, y)$ is differentiable in the open U. Then by definition

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

If $\Delta z = \Delta x + i\Delta y$ and use the form $f(z) = u(x, y) + iv(x, y)$ this is the same as

$$f'(z) = \lim_{\Delta x \to 0, \Delta y \to 0} \frac{\big(u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\big) - \big(u(x, y) + iv(x, y)\big)}{\Delta x + i\Delta y}$$

In particular we can take the limit in the case that $\Delta y = 0$ to get

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{\big(u(x + \Delta x, y) + iv(x + \Delta x, y)\big) - \big(u(x, y) + iv(x, y)\big)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\ &= \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y). \end{aligned}$$

Taking the limit when $\Delta x = 0$ gives

$$\begin{aligned} f'(z) &= \lim_{\Delta y \to 0} \frac{\big(u(x, y + \Delta y) + iv(x, y + \Delta y)\big) - \big(u(x, y) + iv(x, y)\big)}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y} \\ &= \frac{1}{i}\frac{\partial u}{\partial y}(x, y) + \frac{i}{i}\frac{\partial v}{\partial y}(x, y). \\ &= \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y) \end{aligned}$$

Comparing the real and imaginary parts of these two expressions for $f'(z)$ gives

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y), \qquad \frac{\partial v}{\partial x}(x, y) = -\frac{\partial u}{\partial y}(x, y)$$

which are the Cauchy-Riemann equations.