Mathematics 552 Homework due Wednesday, April 12, 2006.

There will be a quiz on Wednesday. Here is what will be on it (view this in part as practice for the test next Monday). Know the following statements.

(1) **Existence of Laurent expansions:** If $0 \leq r < |z - z_0| < R \leq \infty$ and the function $f(z)$ is analytic in the annulus $A = \{z : r < |z - z_0| < R\}$ then $f(z)$ has a convergent ***Laurent expansion***

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

in A.

(2) **Definition of singularity:** The function $f(z)$ has a ***singularity*** at z_0 iff for some $R > 0$ we have that $f(z)$ is analytic in the punctured disk $A = \{z : 0 < |z - z_0| < R\}$. (We will often also that $f(z)$ has an an ***isolated singularity*** at z_0.)

(3) **Existence of Laurent expansions about isolated singularities.** If $f(z)$ has an isolated singularity at z_0 then for some $R > 0$ there is a convergent Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

which holds for $0 < |z - z_0| < R$.

(4) **Classification of singularities:** If $f(z)$ has an isolated singularity at z_0 with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

then the singularity is

(a) ***removable*** iff $a_n = 0$ for all $n \leq -1$. In this case $f(z)$ extends to an analytic function on the disk $\{z : |z - z_0| < R\}$ with power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.

(b) ***a pole*** iff there is a $k \leq -1$ so that $a_n = 0$ for $n < k$ but $a_k \neq 0$. In this case z_0 is a ***pole of order*** $-k$. In the case of $k = -1$ we also call z_0 a ***simple pole***.

Examples: The $f(z) = \dfrac{1}{z}$ has a simple pole (i.e. A pole of order one) at $z_0 = 0$, the function $f(z) = \dfrac{1}{z^k}$ has a pole of order k at $z_0 = 0$, and if $h(z_0) \neq 0$ then $f(z) = \dfrac{h(z)}{(z - z_0)^k}$ has a pole of order k at z_0.

(c) ***essential singularity*** iff there are infinitely many $n \leq -1$ with $a_n \neq 0$.

(5) **Characterization of removable singularities:** Let $f(z)$ have an isolated singularity at z_0. Then the following are equivalent.

(a) $f(z)$ has a removable singularity at z_0.

(b) $f(z)$ is bounded near z_0. That is there is an $R > 0$ and a constant $M > 0$ such that $|f(z)| \leq M$ for $0 < |z - z_0| < R$.

(6) **Characterization of poles:** Let $f(z)$ have an isolated singularity at z_0. Then the following are equivalent.
   (a) $f(z)$ has a pole at z_0.
   (b) $\lim_{z\to z_0} |f(z)| = \infty$.

(7) **Structure of poles of order k:** Let $f(z)$ have an isolated singularity at z_0. Then the following are equivalent.
   (a) z_0 is a pole of $f(z)$ of order k.
   (b) There is a analytic function $h(z)$ defined in a neighborhood of z_0 with

$$f(z) = \frac{h(z)}{(z-z_0)^k}, \qquad \text{and} \qquad h(z_0) \neq 0.$$

(8) **Definition of residue:** If $f(z)$ has an isolated singularity at z_0 and the Laurent expansion of $f(z)$ about z_0 is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

then the ***residue of*** $f(z)$ ***at*** z_0 is

$$\operatorname{Res}(f, z_0) = a_{-1}.$$

That is the residue of $f(z)$ at z_0 is the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion of $f(z)$ at z_0.

(9) **Integrals in circles about singularities:** If $f(z)$ is analytic in $\{z : 0 < |z-z_0| < R\}$ and $0 < r < R$, then

$$\int_{|z-z_0|=r} f(z)\,dz = 2\pi i \operatorname{Res}(f, z_0).$$