Mathematics 552 Homework due Monday, April 10, 2006.

Problem 1. Find the radius of convergence of power series expansion of the following points about the indecated points.

(1) $f(z) = \dfrac{\sin(z)}{z-10}$ about the point $z_0 = 0$.

(2) $f(z) = \dfrac{e^{z^3-4}}{z^2-16}$ about the point $z_0 = 3i$.

(3) $f(z) = \dfrac{z^3-5z+2}{z^2-2z+2}$ about the point $z_0 = 3+4i$.

In doing the following problems you can use either of the following equivalent definitions of z_0 being a zero of $f(z)$ of order k.

Definition 1. If $f(z)$ is analytic in a domain D and $z_0 \in D$, then z_0 is a ***zero of*** $f(z)$ ***of order*** k iff

$$f(z_0) = 0,\ f'(z_0) = 0,\ f''(z_0) = 0, \ldots, f^{(k-1)}(z_0) = 0,\ f^{(k)}(z_0) \neq 0.$$

(That is $f(z)$ and all its derivatives up to order $k-1$ vanish at z_0, but the k-th derivative does not vanish at z_0.)

Definition 2. If $f(z)$ is analytic in a domain D and $z_0 \in D$, then z_0 is a ***zero of*** $f(z)$ ***of order*** k there is an analytic function $h(z)$ in D with $h(z_0) \neq 0$ such that

$$f(z) = (z-z_0)^k h(z).$$

(That is $(z-z_0)^k$ can be factored out of $f(z)$, but no higher power of $(z-z_0)$ can be factored out.)

Problem 2. Show that if $f(z)$ has a zero of order $k \geq 1$ at z_0, then the derivative $f'(z)$ has a zero of order $k-1$ at z_0. HINT: Here it is easiest to use Definition 1.

Problem 3. Show that if $f(z)$ has a zero of order $k \geq 1$ at z_0, the quotient $g(z) = \dfrac{f(z)}{f'(z)}$ has a zero of order 1 at z_0. HINT: This time Definition 2 is probably easier to work with.

Problem 4. If $f(z)$ has a zero of order k at z_0 and $g(z)$ has zero of order ℓ at z_0 then find the order of the zero of the product $p(z) = f(z)g(z)$ at z_0. HINT: This time Definition 2 is probably the easiest to use.

The following is one of the standard results related to the maximum principle and the order of zeros.

Schwartz's Lemma. *Let $D = \{z : |z| < 1\}$ be the unit disk, and let $f(z)$ be analytic in D and continuous in $D \cup \partial D$. Assume that $f(0) = 0$ and $|f(z)| \leq 1$. Then*

$$|f(z)| \leq |z|$$

in D. If equality holds for some $z_1 \neq 0$, then $f(z) = az$ for some constant a with $|a| = 1$.

Problem 5. The following outlines a proof of this result. Assume that $f(z)$ satisfies the hypothesis of Schwartz's Lemma (that is that $f(0) = 0$ and $|f(z)| \leq 1$).

(a) Explain why there is an analytic function $h(z)$ such that $f(z) = zh(z)$. HINT: As $f(0) = 0$ the function $f(z)$ has a zero of order ≥ 1 at $z_0 = 0$. Now one of the two definitions above allows us to factor out a z from $f(z)$. Which definition is this?

(b) As the function $h(z)$ is analytic the maximum of $|h(z)|$ occurs on ∂D, which in our case is the circle $|z| = 1$. (That the maximum of $|h(z)|$ exists is a consequence of the fact that $f(z)$, and therefore $h(z)$, is continuous on the closed bounded set $|z| \leq 1$.) Now justify the following

$$|h(z)| \leq \max_{|z|\leq 1} |h(z)| = \max_{|z|=1} |h(z)| = \max_{|z|=1} \frac{|f(z)|}{|z|} = \max_{|z|=1} \frac{|f(z)|}{1} \leq 1$$

(c) Use $|h(z)| \leq 1$ to show $|f(z)| \leq |z|$.

(d) Finally if equality holds for some $z_1 \in D$ with $z_1 \neq 0$, that if $|f(z_1)| = 1$, then show $|h(z_1)| = 1$ and therefore $|h(z)|$ has an interior maximum at z_1. What does the maximum modulus theorem then say about $h(z)$?

Remark: That assumpition in Schwartz's lemma that $f(z)$ need be continuous on $D \cup \partial D$ can be dropped. The proof only becomes slightly harder.

Extra Credit Problem. What happens in Schwartz's lemma if the hypothesis $f(0) = 0$ and $|f(z)| \leq 1$ are replaced by $f(z)$ has a zero of order $k \geq 1$ at $z = 0$ and $|f(z)| \leq 1$. Prove your result. HINT: Start with $f(z) = z^k h(z)$ and proceed as above.