Mathematics 552 Homework due Monday, April, 2006.

(1) Text Problem 1, Page 125.
(2) Text Problem 2, Page 125. Hint: As $f(z_0) \neq 0$ the function $h(z) = 1/f(z)$ is analytic near z_0.
(3) Text Problem 3, Page 125.
(4) Text Problem 4, Page 125.
(5) Text Problem 5, Page 125.
(6) (Extra Credit). With the notation of the last problem, let $f(z)$ be an analytic function such that $M(r) \leq A + Br$ for some positive constants A and B. Show that $f(z)$ is a polynomial of degree $\leq a$. Hint: To show that $f(z)$ is a polynomial of degree at most one, it is enough to show that $f'' \equiv 0$. We have a formula

$$f''(a) = \frac{2}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^3}\, dz.$$

Assume that $R > |a|$. On the circle $|z - a| = R$ explain why

$$|z| \leq |a| + R$$

and thus on $|z - a| = R$ that

$$|f(z)| \leq M(|a| + R) \leq A + B(|a| + R).$$

Also on $|z - a| = R$

$$|z - a| \geq R - |a|$$

and therefore

$$\frac{1}{|z-a|^3} \leq \frac{1}{(R - |a|)^3}.$$

Thus on $|z - a| = R$

$$\left| \frac{f(z)}{(z-a)^3} \right| \leq \frac{A + B(|a| + R)}{(R - |a|)^3}.$$

Use this in the formula for $f''(a)$ to get

$$|f''(a)| \leq \frac{1}{\pi} \frac{A + B(|a| + R)}{(R - |a|)^3} \operatorname{Length}(|z - a| = R).$$