Mathematics 552 Homework due Monday, March 27, 2006.

As stated on the last homework, the goal for a while is to use the Cauchy integral formula to deduce facts about analytic functions. In this set of problems we look at the natural case of what happens when we apply the Cauchy Integral Formula to a disk and use it to compute the value at the center of the disk. If you are adventuresome you might try doing this before looking at the theorem and problems below.

Theorem 1 (Mean Value Property of Analytic Functions)**.** *Let $f(z)$ be analytic in the domain D, and assume that the closed disk with center a and radius r (that is $\{z : |z-a| \le r\}$) is contained in D. Then*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt. \tag{1}$$

(This can be loosely restated as: The average of an analytic function on a circle is equal to the value of the function at the center of the circle.)

Problem 1. Prove this by doing the following.

(a) Show (and this is not much more that a restatement of the Cauchy Integral Formula for a disk)

$$f(a) = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{z-a}\,dz.$$

(b) In this integral parameterize the circle $|z-a| = r$ by $z = a + re^{i}t$ and do the substitution. This should lead to the desired result. □

Remark: It use to be standard terminology to refer to the "average value" as the "mean value" (and still is in statistics and probability). This is where the term "mean value property" comes from.

Problem 2. The mean value property is sometime written as follows. With the same hypothesis as in Theorem 1

$$f(a) = \frac{1}{2\pi r}\int_{|z-a|=r} f(z)\,ds \tag{2}$$

where ds is arclength along the circle $|z-a| = r$. (Note that $2\pi r$ is the length of the circle $|z-a| = r$ so that this is still expressing $f(a)$ as the average value of $f(z)$ over the circle $|z-a| = r$.) Show that this follows from Theorem 1 by a change of variable as follows

(a) Let $z = a + re^{it}$. Then we know that $ds = |z'(t)|\,dt$. What is $|z'(t)|\,dt$ in this case?

(b) To the change of variable $z = a + re^{it}$ in equation (1) to deduce equation (2) holds. □

Integration Bee.

Here a copy of an an e-mail from Carrie Finch:

I was walking down the hall this afternoon, and the Integration Bee snuck up behind me and scared the heck out of me. Just so you don't get taken by surprise too, I wanted to warn you all that the Integration Bee is lurking just around the corner. Here are the details:

Integration Bee - open to all students, visitors & faculty
Lots of prizes!!
Tuesday, 28 March 2006, 7:30pm (until about 9pm)
LeConte - Room 412

We'll have snacks and drinks; feel free to bring your favorite snacks along with you. This year will again be a team competition with music, food, and fun! Be sure to tell your classes about it!

See you there!